Modified Laplace-Beltrami Quantization *of* Extended Systems *with* Quadratic Constants of Motion

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Superintegrable systems

Classical: H Hamiltonian with n degrees of freedom
Quantum: $\hat{H}$ Hamiltonian differential operator with n degrees of freedom

Classical

H is maximally superintegrable if it has the maximal number of functionally independent constants of the motion $= 2n - 1 \rightarrow$ finite orbits are closed (or their closure is 1-dimensional)

Quantum

$\hat{H}$ is maximally superintegrable if it has the maximal number of algebraically independent symmetry operators $= 2n - 1 \rightarrow$ complete degeneration of the energy levels

Typical classical example:

The anisotropic two-dimensional oscillator of Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \omega_1 x^2 + \omega_2 y^2$$

is an integrable system for all values of ω_1, ω_2:

- it is superintegrable if the ratio ω_1/ω_2 is rational (it admits a third global first integral and orbits are Lissajou curves)
- it is not superintegrable if the ratio ω_1/ω_2 is irrational (orbits fill a rectangle in (x, y)-plane).

Laplace-Beltrami quantization

Quantization:

$$\{H, K\} = 0 \to [\hat{H}, \hat{K}] = 0$$

Laplace-Beltrami quantization

Hamiltonian

$$H = \frac{1}{2} g^{ij} p_i p_j + V \ \to \ \hat{H}\psi = \left(-\frac{\hbar^2}{2} g^{ij} \nabla_i \nabla_j + V\right)\psi,$$

Quadratic first integrals

$$K = \frac{1}{2} K^{ij} p_i p_j + V_K \to \hat{K}\psi = \left(-\frac{\hbar^2}{2} \nabla_i K^{ij} \nabla_j + V_K\right)\psi,$$

If the curvature is not constant

$$\{H, K\} = 0 \nrightarrow [\hat{H}, \hat{K}] = 0$$

Modified LB-quantization

Hamiltonian

$$H = \frac{1}{2} g^{ij} p_i p_j + V \to \hat{H}_E \psi = \left(-\frac{\hbar^2}{2} (g^{ij} \nabla_i \nabla_j + E) + V \right) \psi,$$

Quadratic first integrals

$$K = \frac{1}{2} K^{ij} p_i p_j + V_K \to \hat{K}_{E_K} \psi = \left(-\frac{\hbar^2}{2} (\nabla_i K^{ij} \nabla_j + E_K) + V_K \right) \psi,$$

scalars E and E_K are quantum corrections. They may depend on the scalar curvature and the squared Weyl scalar $\sqrt{3 W_{abcd} W^{abcd}}$

Main example

Quadratic first integrals of extended Hamiltonian can be quantized by modified LB quantization.

Modified LB-Quantization

Let Δ_E be the modified Laplacian $g^{ij}\nabla_i\nabla_j + E$, let be $\hat{H}_E = -\frac{\hbar^2}{2}\Delta_E + V$ and $\hat{K}_{E_K} = -\frac{\hbar^2}{2}(\Delta_K + E_K) + V_K$, where E_K is a scalar to be determined, then

Theorem

$$[\hat{H}_E, \hat{K}_{E_K}] = 0$$

if and only if

1. $\mathbf{K}$ is a Killing tensor,
2. the following equation holds

$$\frac{\hbar^2}{6}\delta(\mathbf{KR} - \mathbf{RK}) + \frac{\hbar^2}{2}\left(\nabla E_K - \mathbf{K}\nabla E\right) + K\nabla V - \nabla V_K = 0.$$

δ is the divergence operator, $\mathbf{R}$ is the Ricci tensor.

K is a first integral of H if and only if $\mathbf{K}$ is a Killing tensor and

$$\mathbf{K}\nabla V - \nabla V_K = 0.$$

Therefore,

Theorem

If $\{H, H_K\} = 0$ then $[\hat{H}_E, \hat{K}_{E_K}] = 0$ if and only if

$$\nabla E_K + \frac{1}{3}\delta(\mathbf{KR} - \mathbf{RK}) - \mathbf{K}\nabla E = 0.$$

From which integrability conditions for E_K can be immediately obtained

The integrability conditions for E_K are

$$\nabla_i Z_j - \nabla_j Z_i = 0,$$

with $Z_i = (\frac{1}{3}\delta(\mathbf{KR} - \mathbf{RK}) - \mathbf{K}\nabla E)_i$, $i, j = 1, \ldots, N$.

MLBQ for Stäckel systems

Stäckel systems $(H, K_2, \dots, K_n)$ *(= H-J separation of variables, n quadratic first integrals in Poisson involution, commuting as linear operators)*

if $\delta(\mathbf{K}_i\mathbf{R} - \mathbf{R}\mathbf{K}_i) = 0 \to$ LBQ for all quadratic f.i. in involution

Superintegrable Stäckel systems may need MLBQ

$(H, K_2, \dots, K_n)$ Stäckel system, E a QC for H, then

if $\delta(\mathbf{K}_i\mathbf{R} - \mathbf{R}\mathbf{K}_i) = 0$ and $\nabla(\mathbf{K}_i\nabla E) = 0$
$\to (H - \frac{\hbar^2}{2}E, K_2 - \frac{\hbar^2}{2}E_2, \dots, K_n - \frac{\hbar^2}{2}E_n)$ Stäckel system
$\to (\hat{H}_E, \dots \hat{K}_{E_n})$ MLBQ integrable

$\nabla E_i = \mathbf{K}_i\nabla E$ hence $K_i + E_i$ and $H + E$ are still a Stäckel system.

Extended Systems with Quadratic first integrals

Let $L(q^i, p_i)$ be a natural Hamiltonian s.t. $\{L, \{L, G(q^i)\}\} = -2(cL + L_0)G(q^i)$ $(c, L_0 \in \mathbb{R})$ then $\{H, K\} = 0$ where

$$
\begin{aligned}
H &= \frac{1}{2}p_u^2 - 4\gamma' L + 4L_0\gamma^2 + \frac{1}{2}\omega\gamma^{-2}, \\
K &= p_u^2 G + 4\gamma X_L G p_u + 4\gamma^2 X_L^2 G + \frac{\omega}{\gamma^2} G \\
&= p_u^2 G + 4\gamma \nabla^i G p_i p_u - 4c\gamma^2 G g^{ij} p_i p_j - 8\gamma^2 (cV + L_0) G + \frac{\omega}{\gamma^2} G
\end{aligned}
$$

$$
\gamma = \begin{cases} -Au \\ \dfrac{C_\kappa(cu)}{S_\kappa(cu)} \end{cases} \qquad \gamma' = \begin{cases} -A & c = 0, \\ \dfrac{-c}{S_\kappa^2(cu)} & c \neq 0, \quad \kappa = A/c, \end{cases}
$$

$$
S_\kappa(x) = \begin{cases} \frac{\sin\sqrt{\kappa}x}{\sqrt{\kappa}} & \kappa > 0 \\ x & \kappa = 0 \\ \frac{\sinh\sqrt{|\kappa|}x}{\sqrt{|\kappa|}} & \kappa < 0 \end{cases} \qquad C_\kappa(x) = \begin{cases} \cos\sqrt{\kappa}x & \kappa > 0 \\ 1 & \kappa = 0 \\ \cosh\sqrt{|\kappa|}x & \kappa < 0 \end{cases}
$$

are known as Trigonometric Tagged Functions

H is a particular case with $m/n = 2$ of a family of Hamiltonians depending on a rational parmeter which admit a first integral constructed by a recursion method.
Let be $c,\ L_0 \in \mathbb{R},\ \gamma(u),\ G(q^i, p_i)$ such that

$$X_L^2 G = -2(cL + L_0)G, \qquad \gamma' + c\gamma^2 + A = 0$$

$$H_{m,n} = \frac{1}{2}p_u^2 - \frac{m^2}{n^2}\gamma' L + \frac{m^2}{n^2} L_0 \gamma^2 + \omega\gamma^{-2}, \qquad \omega \in \mathbb{R},\ m, n \in \mathbb{N}^*.$$

$$K_{2s,k} = \left((p_u + 2s/k^2 \gamma X_L)^2 + 2\omega\gamma^{-2}\right)^s G_k. \qquad s, k \in \mathbb{N}^*$$

$$G_1 = G, \qquad G_{n+1} = X_L(G)\, G_n + \frac{1}{n} G\, X_L(G_n),$$

Theorem [CDR 2014]

$$\{H_{m,n}, K_{m,n}\} = 0 \qquad \text{for } m = 2s.$$

$$\{H_{m,n}, K_{2m,2n}\} = 0 \qquad \text{for } m = 2s + 1.$$

Geometry underlying extensions

When L is a natural Hamiltonian, the metric on the configuration manifold of H is a warped metric

Warped metrics

Let (M, g^M, x^i) and (N, g^N, y^a) be Riemannian metrics, their warped product is a metric on $N \times M$ of the form

$$ds^2 = g^N_{ab}(y^c)\, dy^a\, dy^b + f(y^c) g^M_{ij}(x^k)\, dx^i\, dx^j$$

For $dim(N) = 1$:

$$ds^2 = dy^2 + f(y) g^M_{ij}(x^k)\, dx^i\, dx^j$$

$\{L, \{L, G(q^i)\}\} = -2(cL + L_0)G(q^i) \Leftrightarrow$

$$\nabla^i \nabla^j G + c g^{ij} G = 0 \qquad (1)$$

$$\nabla_i V \nabla^i G - 2(cV + L_0)G = 0.$$

$G(q^i)$ may depend on up to $n+1$ parameters a_i (complete),

Theorem

(1) has a complete solution $\Leftrightarrow$ g is of constant curvature c.

These are the only non trivial solutions on a two dimensional manifold.
Less complete solutions can be found on manifolds of non-constant curvature of dimension ≥ 3.

Remark

Equation (1) is fundamental in theory of warped metrics (Yoshihiro Tashiro 1965). $\Rightarrow$ the base manifold of L admits a warped metric.

- If L has $n = 1$ degree of freedom, then its extension H is based on a constant curvature two-dimensional manifold.
- If L has $n \geq 2$ degrees of freedom and is based on a constant curvature manifold, its extension H is based on a conformally flat manifold, not of constant curvature.
- If L has $n > 2$ degrees of freedom and is based on a non-constant curvature manifold, its extension H can be based on a non conformally flat manifold.

Remark

By repeating at least twice the extension procedure we drop on non constant curvature manifolds and (possibly) on non conformally flat manifolds.

Classical Laplace-Beltrami quantization does not produce commuting operators. [CDR2011]

Genesis of extensions: Interesting three-body-on-a-line systems

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + V$$

Jacobi-Calogero: two-body interaction

$$V_C = \frac{b}{(x-y)^2} + \frac{b}{(y-z)^2} + \frac{b}{(z-x)^2}.$$

Wolfes: three-body interaction

$$V_W = \frac{b}{[(x-y)-(y-z)]^2} + \frac{b}{[(y-z)-(z-x)]^2} + \frac{b}{[(z-x)-(x-y)]^2}.$$

The Jacobi-Calogero and Wolfes systems are separable in all rotational Stäckel systems with axis parallel to $\omega = (1,1,1)$ through the origin CDR JMP 49 (2008): Circular cylindrical, Spherical, Paraboloidal, Spheroidal Prolate and Oblate.

Both are maximally superintegrable with four quadratic and one CUBIC in the momenta first integrals.
In Cylindrical coordinates (r, ψ, w)

$$H = \frac{1}{2}(p_r^2 + \frac{1}{r^2} p_\psi^2 + p_w^2) + V$$

Jacobi-Calogero:

$$V_C = \frac{a}{r^2 \sin^2(3\psi)},$$

Wolfes:

$$V_W = \frac{a}{r^2 \cos^2(3\psi)}.$$

They coincide after a rotation of $\frac{\pi}{6}$ in $\mathbb{E}^3$.

Both show a dihedral (hexagonal) symmetry hidden in the original coordinate system.
Is the hexagonal symmetry responsible for the cubic first integral making maximally superintegrable the systems?

Conjecture

$$H = \frac{1}{2}p_r^2 + \frac{1}{r^2}\left(\frac{1}{2}p_\psi^2 + \frac{a}{\sin^2 k\psi}\right)$$

is Superintegrable for $k \in \mathbb{N}$, with a first integral of degree k, for $k = \frac{1}{n}$ with a polynomial first integral of degree at least n
CDR JMP 49 (2008)

Generalized conjecture

$$H = \frac{1}{2}p_r^2 + \omega^2 r^2 + \frac{1}{r^2}\left(\frac{1}{2}p_\psi^2 + \frac{a}{\sin^2 k\psi} + \frac{b}{\cos^2 k\psi}\right)$$

$k \in \mathbb{Q}$ is Superintegrable
Tremblay, Turbiner, Winternitz J. Phys. A 42 (2009)

Theorem

YES!
Kalnins, Kress, Miller SIGMA 6 (2010)

The previous systems have Hamiltonians of type

$$H = \frac{1}{2}p_u^2 + \alpha(u)\left(\frac{1}{2}g^{ij}p_ip_j + V(q^i)\right) + \beta(u)$$

Problem

Determine $\alpha(u)$, g, V, $\beta(u)$ such that H admits first-integral of high degree, effectively computable, without assuming separation of variables

A solution is...

If there exist constants c, L_0 such that $X_L^2 G = -2(cL + L_0)G$ admits a solution $G(q^i, p_i)$, then

$$H = \frac{1}{2}p_u^2 - \frac{m^2}{n^2}\gamma' L + \frac{m^2}{n^2}L_0\gamma^2 + \omega\gamma^{-2}, \qquad \omega \in \mathbb{R}, m, n \in \mathbb{N}^*.$$

$$K_{2s,k} = \left((p_u + 2s/k^2\gamma X_L)^2 + 2\omega\gamma^{-2}\right)^s G_k. \qquad s, k \in \mathbb{N}^*.$$

are in involution for $2s/k = m/n$
where $G_1 = G, \qquad G_{n+1} = X_L(G)\, G_n + \frac{1}{n} G\, X_L(G_n),$

$$\gamma = \begin{cases} -Au \\ \\ \dfrac{C_\kappa(cu)}{S_\kappa(cu)} \end{cases} \qquad \gamma' = \begin{cases} -A & c = 0, \\ \\ \dfrac{-c}{S_\kappa^2(cu)} & c \neq 0, \quad \kappa = A/c, \end{cases}$$

$$S_\kappa(x) = \begin{cases} \frac{\sin\sqrt{\kappa}x}{\sqrt{\kappa}} & \kappa > 0 \\ x & \kappa = 0 \\ \frac{\sinh\sqrt{|\kappa|}x}{\sqrt{|\kappa|}} & \kappa < 0 \end{cases} \qquad C_\kappa(x) = \begin{cases} \cos\sqrt{\kappa}x & \kappa > 0 \\ 1 & \kappa = 0 \\ \cosh\sqrt{|\kappa|}x & \kappa < 0 \end{cases}$$

Ex.

$$
\begin{aligned}
L &= \frac{1}{2}p_\phi^2 + \frac{a}{\sin^2\phi}, \\
G &= \cos\phi, \qquad c = 1,\ L_0 = 0 \\
H &= \frac{1}{2}p_r^2 + \frac{m^2}{r^2}L = \frac{1}{2}p_r^2 + \frac{1}{r^2}\left(\frac{1}{2}p_\psi^2 + \frac{m^2 a}{\sin^2 m\psi}\right), \quad \phi = m\psi, \\
K &= \left(p_r + \frac{m}{r}(p_\phi\partial_\phi - \partial_\phi V\partial^\phi)\right)^m G.
\end{aligned}
$$

$m = 3$ Jacobi-Calogero system; K is CUBIC first integral.

Other examples are (caged) anisotropic oscillators and the TTW systems, where G is linear in the momentum p_ϕ.

Theorem

If the Hamiltonian L is superintegrable with $2n-1$ first integrals, then its extension H is superintegrable with $(2n-1)+2 = 2(n+1)-1$ first integrals.

we add H and K constructed through the recursion.

Back to Quantum corrections

We look for scalars E and E_K to associate to the classical commuting Hamiltonians

$$
\begin{aligned}
H &= \frac{1}{2}p_u^2 - 4\gamma' L + 4L_0\gamma^2 + \frac{1}{2}\omega\gamma^{-2}, \\
K &= p_u^2 G + 4\gamma\nabla^i G p_i p_u - 4c\gamma^2 G g^{ij} p_i p_j - 8\gamma^2(cV + L_0)G + \frac{\omega}{\gamma^2}G
\end{aligned}
$$

commuting operators $\hat{H}_E = -\frac{\hbar^2}{2}(\Delta + E) + V$
$\hat{K}_{E_K} = -\frac{\hbar^2}{2}(\Delta_K + E_K) + V_K$.
Let us denote
$Sc =$ scalar curvature of the metric $\mathbf{g}$ of L ($Sc = g^{ij}R_{ij}$, R_{ij} Ricci)
$\widetilde{Sc} =$ scalar curvature of the extended metric tensor $\widetilde{\mathbf{g}}$ of H

Theorem

The scalar functions

$$E = \frac{1-n}{4n}\widetilde{Sc} = \frac{n-1}{4n}\gamma'(4Sc + cn(n-1)) + \frac{n^2-1}{4}cA,$$

$$E_K = c(n-1)G\left(\frac{\gamma^2}{2n}(4Sc + cn(3n-7)) - A\right).$$

are quantum corrections s.t. the modified LB operators $\widehat{H}_E$ and $\widehat{K}_{E_K}$ commute.

- The modified Laplacian Δ_E is the conformally invariant Laplacian of $\tilde{\mathbf{g}}$.
- If L has more quadratic first integrals, is it possible to find quantum correction allowing the simultaneous commutation?

MLBQ of quadratic first integrals of L

— extension of Stäckel systems, iterated extensions—

Assume L has a quadratic first integral J admitting LB quantization.
Let H be the extension of L

$$H = \frac{1}{2}p_u^2 - 4\gamma' L + 4L_0\gamma^2 + \frac{1}{2}\omega\gamma^{-2},$$

with the quadratic first integral

$$K = \left((p_u + 2\gamma X_L)^2 + \omega\gamma^{-2}\right) G.$$

Prop 1

$\{L, J\} = 0, \qquad \delta(\mathbf{RJ} - \mathbf{JR}) = 0 \qquad \nabla(\mathbf{J}\nabla(Sc)) = 0$
$\rightarrow \exists E_J$ defined by $\nabla E_J = \frac{1-n}{4n}\mathbf{J}\nabla Sc$ i.e., $\{H + E, J + E_J\} = 0$
determines simultaneous MLBQ for K and J

$$[\hat{H}_E, \hat{J}_{E_J}] = [\hat{H}_E, \hat{K}_{E_K}] = 0$$

MLBQ of Extensions of Hamiltonians with their own MLBQ

Assume L admits quadratic first integrals $J^1, \dots, J^k$ with the simultaneous MLBQ

$$\hat{L}_{\bar{E}_L}, \hat{J}^1_{\bar{E}_1}, \dots, \hat{J}^k_{\bar{E}_k}$$

Consider the extension of L

$$H = \frac{1}{2}p_u^2 - 4\gamma' L + 4L_0\gamma^2 + \frac{1}{2}\omega\gamma^{-2},$$

with the quadratic first integral $K = \left((p_u + 2\gamma X_L)^2 + \omega\gamma^{-2}\right) G$.

Prop 2

$[\hat{L}_{\bar{E}_L}, \hat{J}^i_{\bar{E}_i}] = 0$, $\nabla(\mathbf{J}^i\nabla(Sc)) = 0$ and $\nabla_i \bar{E}_L \nabla^i G = 2cG\bar{E}_L$
$\rightarrow \tilde{E} = E - 4\gamma' \bar{E}_L, \qquad \tilde{E}_K = E_K - 8c\gamma^2 G\bar{E}_L$
determine MLBQ

$$[\hat{H}_{\tilde{E}}, \hat{K}_{\tilde{E}_k}] = 0, \quad [\hat{H}_{\tilde{E}}, \hat{J}^i_{\bar{E}_i + E_i}] = 0$$

Example on Non-Constant-Curvature manifold

E G Kalnins, J M Kress and W Miller Jr (2013)
Superintegrability in a non-conformally-flat space

$$
\begin{aligned}
H = L_4 = p_r^2 + \alpha r^2 + \frac{L_3}{r^2} \\
L_3 = p_{\theta_1}^2 + \frac{\beta_1}{\cos^2(k_1\theta_1)} + \frac{L_2}{\sin^2(k_1\theta_1)} \\
L_2 = p_{\theta_2}^2 + \frac{\beta_2}{\cos^2(k_2\theta_2)} + \frac{L_1}{\sin^2(k_2\theta_2)} \\
L_1 = p_{\theta_3}^2 + \frac{\beta_3}{\cos^2(k_3\theta_3)} + \frac{\beta_4}{\sin^2(k_3\theta_3)}
\end{aligned}
$$

Maximally Superintegrable classically and quantum for $k_i \in \mathbb{Q}$
and Iterated Extension in Non-Conformally Flat Space for $k_i \in \mathbb{Q}$
We call K_i the first integrals associated with the extension procedures.

Modified LB Quantization

$$\hat{H} = \hat{L}_4 = \Delta + V - \frac{1}{6} Sc_4 - \frac{1}{24} W_4$$

Scalar Curvature

$$Sc_4 = \frac{6}{r^2} + k_1^2 \left(\frac{6}{r^2} - \frac{2}{r^2 \sin^2(k_1 \theta_1)} \right) + \frac{2 k_2^2}{r^2 \sin^2(k_1 \theta_1)}$$

"Weyl scalar"

$$W_4 = \sqrt{3 W_{abcd} W^{abcd}} = \frac{2(k_1^2 - k_2^2)}{r^2 \sin^2(2\theta_1)}$$

Conformally Invariant Laplacian + an additional correction which vanishes on conformally flat manifolds.

In the case of all quadratic first integrals (i.e., $k_i = 2^i$), with the rescaling $u^i = k_i \theta_i$)

$$H = \frac{1}{2} p_r^2 + \frac{4}{r^2} \left(\frac{1}{2} p_1^2 + \frac{4}{sin^2 u^1} \left(\frac{1}{2} p_2^2 + \frac{4}{sin^2 u^2} \left(\frac{1}{2} p_3^2 \right) \right) \right)$$

the term $E_4 = -\frac{1}{6} Sc_4$ is the one needed to quantize K_4 and W_4 is explained by Prop 2:
$\hat{L_3}_{\bar{E}_3}, \hat{K_3}_{\bar{E}_K}$ are MLBQ of L_3, K_3 with $\bar{E}_3 = -1/8 Sc_3$ the hypotheses of Prop. 2 hold: $\nabla(K_3 \nabla(Sc_3)) = 0$, $\nabla_i \bar{E}_3 \nabla^i G_4 = 2c G_4 \bar{E}_3$

$$-\frac{1}{6} Sc_4 - \frac{1}{24} W_4 = E_4 - 4\gamma' \bar{E}_3 = E_4 - \frac{1}{2} \gamma' Sc_3$$

(L_2, L_1 and K_2 do not add quantum corrections, since Sc_2 is constant)

Conclusions

1. The extension procedure is a method for building a class of Hamiltonians with one high-order first integral (ex: TTW system)
2. Connected with conformal symmetries and warped manifolds
3. Extensions of superintegrable systems are superintegrable systems
4. When the first integrals generated by the extension procedure are quadratic, we developed a modified Laplace-Beltrami quantization (Conformally Invariant Laplacian)
5. When L has quadratic first integrals (ex: Separation of Variables), we have necessary and sufficient conditions for the simultaneous MDLQ of all the integrals.

Open Problems:

6. How to quantize first integrals for $m > 2$
7. Links with Kuru-Negro SUSY quantization

Thank you for your attention!