Straightenings of Lie supergroups

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Abstract

After a quick overview of a differential-geometric approach to supermanifolds, the main result of which is a flowbox theorem for these spaces, I'll focus on applying this approach to the theory of Lie supergroups. In particular we'll talk about "straightenings", understood as isomorphisms of an arbitrary supermanifolds with a split one.

Smooth supermanifolds and their morphisms

A **smooth supermanifold of superdimension** $(m|n)$ is an object $(M|\mathcal{R}M)$ such that

1. M is a smooth manifold of dimension m;
2. $\mathcal{R}M$ is a smooth bundle of free supercommutative unital algebras pointwise isomorphic to ΛS^*, where $\dim S = n$.

A **smooth supermap** $(\phi|\Phi):\ (M|\mathcal{R}M) \to (N|\mathcal{R}N)$ consists of

- a smooth map $\phi\colon M \to N$
- and a unital homomorphism of superalgebras $\Phi\colon \Gamma(\mathcal{R}N) \to \Gamma(\mathcal{R}M)$.

The smooth supermaps whose codomain is a linear supermanifod $(V|S)$ are very simple:

Theorem - *The map Φ is a **differential operator along** ϕ under the commutator*

$$[\Phi_\phi f](\eta) = \Phi(f\eta) - (f \circ \phi)\Phi(\eta)$$

for $f \in \mathcal{C}^\infty(N)$ and $\eta \in \Gamma(\mathcal{R}N)$.

Theorem - *Let $\mathcal{C}^\infty((M|\mathcal{R}M),(V|\mathbf{S}))$ denote the space of supersmooth maps. There are isomorphisms of superalgebras*

$$\begin{aligned}\mathcal{C}^\infty((M|\mathcal{R}M),(V|\mathbf{S})) &\cong \mathrm{Hom}_{\mathrm{alg}}\left(\mathcal{C}^\infty(V,\Lambda_\bullet\mathbf{S}^*),\Gamma(\mathcal{R}_\bullet M)\right)\\ &\cong \mathrm{Hom}_{\mathrm{alg}}\left(\mathrm{Sym}\,V^*\otimes\Lambda_\bullet\mathbf{S}^*,\Gamma(\mathcal{R}_\bullet M)\right)\\ &\cong V\otimes\Gamma(\mathcal{R}_+M)\oplus\mathbf{S}\otimes\Gamma(\mathcal{R}_-M)\end{aligned}$$

where $\mathrm{Hom}_{\mathrm{alg}}$ denotes superalgebra homomorphisms.

The tangent and cotangent superbundles

Recall that the derivations of $\mathcal{C}^\infty(M)$ are sections of the vector bundle TM.

For every point p of M define the space

$$\mathrm{sder}_p(\mathcal{R}M) = \mathrm{sder}(\mathcal{R}_pM)$$

of **pointwise derivations** of $\mathcal{R}M$. The vector bundle

$$\mathrm{sder}(\mathcal{R}M) := \sqcup_{p\in M}\mathrm{sder}_p(\mathcal{R}M).$$

is the **bundle of pointwise derivations** of $(M|\mathcal{R}M)$.

Theorem - *The following sequence of vector bundles is exact*

$$0 \to \mathrm{sder}\,\mathcal{R}M \hookrightarrow \mathrm{Der}\,\mathcal{R}M \xrightarrow{\sigma} \mathcal{R}M\otimes TM \to 0$$

Here σ denotes the principal symbol.

Corollary - *The bundle $\mathrm{Der}\,\mathcal{R}M$ is generated as a left $\mathcal{R}M$-module by $TM\oplus\mathbf{S}M$.*

Tangential maps

Lemma - *Given a supersmooth map $(\phi|\Phi):(M|\mathcal{R}M)\to(N|\mathcal{R}N)$ there are two bundle maps $F\colon\mathbf{S}M\to\mathbf{S}N$ and $F^*\colon\mathbf{S}^*N\to\mathbf{S}^*M$ naturally associated to it.*

The **differential** and **codifferential** of $(\phi|\Phi)$ are

$$(\phi|\Phi)_* = (\phi|F) : TM\oplus\mathbf{S}M \to TN\oplus\mathbf{S}N$$
$$(\phi|\Phi)^* = (\phi|F^*) : T^*N\oplus\mathbf{S}^*N \to T^*M\otimes\mathbf{S}^*M$$

Let $(\phi|\Phi):(M|\mathcal{R}M)\to(N|\mathcal{R}N)$ be a supersmooth map. The **auxiliary codifferential** of the supersmooth map $(\phi|\Phi)$ is the map $\Phi^!$ defined by

$$\Phi^! : T^*N \to \Lambda^2\mathbf{S}^*M = \mathcal{R}^{\geqslant 2}M/\mathcal{R}^{\geqslant 3}M$$
$$df_{\phi(p)} \mapsto \Phi(f)_p - (f\circ\phi)(p) + \mathcal{R}^{\geqslant 3}_pM$$

The **auxiliary differential** is the dual map $\Phi_! := (\Phi^!)^*$.

Lemma - *If the map $\Phi:\Gamma(\mathcal{R}N)\to\Gamma(\mathcal{R}M)$ is a differential operator of order 0 then $\Phi_!\equiv 0$.*

Straightenings

***Theorem** (Flowbox coordinates for supermanifolds)* - *There is a bijection between the set*

$$\mathcal{S}(M|\mathcal{R}M) = \{\psi|\psi\colon(TM\oplus\mathbf{S}M)_\bullet\hookrightarrow\mathrm{Der}_\bullet(\mathcal{R}M)\}$$

of $\mathbb{Z}_2$-graded inclusions of the bundle of generators $TM\oplus\mathbf{S}M$ into the bundle of superderivations $\mathrm{Der}(\mathcal{R}M)$, and the set $\{\Psi\colon\Lambda\mathbf{S}^*M\xrightarrow{\cong}\mathcal{R}M\}\times\{\nabla|\nabla$ connection on $\mathbf{S}M\}$ of bundle isomorphisms and connections satisfying the following properties:

1. *if $s,\tilde{s}$ are sections of $\mathbf{S}M$ then $[\![\psi(s),\psi(\tilde{s})]\!]=0$;*
2. *for any $s\in\Gamma(\mathbf{S}M)$ and any vector field X on M there is a unique $\tilde{s}$ such that $[\![\psi(X),\psi(s)]\!]=\psi(\tilde{s})$;*

the bijection is realized in the following way: if s is a section of $\mathbf{S}M$ and σ a section of $\Lambda\mathbf{S}^*M$ then $\Psi(s\lrcorner\sigma)=\psi(s)(\Psi(\sigma))$.

$\mathbb{Z}$-graded Lie supergroups

A supermanifold for which $\mathcal{R}M=\Lambda\mathbf{S}^*M$, denoted $(M|\mathbf{S}M)$, is called **split or $\mathbb{Z}$-graded**.

Theorem - *The group objects in the category of $\mathbb{Z}$-graded supermanifolds are exactly the **bihomogeneous vector bundles** $\mathbf{S}G$ over a Lie group G.*

Let S be a representation space for a Lie group G. Every bihomogeneous vector bundle is of the form

$$\mathbf{S}G = G\times S\times G\big/\langle(g,s,h)\sim(gk^{-1},k\cdot s,kh)|k\in G\rangle$$

There is a **biaction** $G\times\mathbf{S}G\times G\to G$ that permutes fibres and acts linearly by $g*s*h=(g,h^{-1})\cdot s$. Note that the multiplication map $m\colon G\times G\to G$ allows one to pullback the bundle $\Lambda(\mathbf{S}^*G\oplus\mathbf{S}^*G)$ to $G\times G$. Observe that any Lie supergroup whose base is the Lie group $G\times G$ has as bihomogeneous vector bundle $\Lambda(\mathbf{S}^*G\boxplus\mathbf{S}^*G)$. There is a natural biequivariant isomorphism

$$\mathrm{twist}\colon m^*\big(\Lambda(\mathbf{S}^*G\oplus\mathbf{S}^*G)\big)\xrightarrow{\cong}\Lambda(\mathbf{S}^*G\boxplus\mathbf{S}^*G)$$
$$s_{xy}\oplus\tilde{s}_{xy}\mapsto(e*s_{xy}*y^{-1})\oplus(x^{-1}*s_{xy}*e)$$

that "twists up" the fibres by forgetting the product; indeed the right hand size is on $\Lambda\mathbf{S}^*_xG\otimes\Lambda\mathbf{S}^*_yG$.

Lemma - *There are two natural trivializations of $\mathbf{S}G\cong G\times S=G\times\mathbf{S}_eG$ given by*

$$\tau^L(s_g)=g^{-1}*s_g*e$$
$$\tau^R(s_g)=e*s_g*g^{-1}$$

which give rise to two natural connections ∇^L and ∇^R on $TG\oplus\mathbf{S}G$.

Parallels with symmetric spaces

Theorem - *A Lie superalgebra $\mathcal{L}=(\mathfrak{g}|\mathbf{S})$ is completely determined by the following data:*

- *A representation ρ of the Lie algebra $\mathfrak{g}$ on the vector space $\mathbf{S}$;*
- *A symmetric $\mathfrak{g}$-equivariant bilinear map $B\colon\mathrm{Sym}^2\mathbf{S}\to\mathfrak{g}$ that lies in the kernel of the composition*

$$\mathrm{Sym}^2\mathbf{S}^*\otimes\mathfrak{g}\xrightarrow{\mathrm{id}\otimes\rho}\mathrm{Sym}^2\mathbf{S}\otimes\mathbf{S}^*\otimes\mathbf{S}\xrightarrow{f\otimes\mathrm{id}}\mathrm{Sym}^3\mathbf{S}^*\otimes\mathbf{S}$$

where $f(\alpha\otimes\sigma)=\alpha\cdot\sigma$.

On a symmetric space K/G the Lie algebra is $\mathbb{Z}_2$-graded: $\mathfrak{k}=\mathfrak{g}\oplus\mathfrak{p}$ and the bracket decomposes as above, with B alternating; also for all $X,Y\in\mathfrak{p}$:

$$e^Xe^Y=e^{k(X,Y)}e^{p(X,Y)}$$

where $p,k\in\left[\overline{\mathrm{Sym}}(\mathfrak{p}^*\oplus\mathfrak{p}^*)\otimes\mathfrak{k}\right]^G$ are power series in the bracket (p collects the terms with odd brackets and k the even ones) and

$$\mathcal{C}^\infty(K)\approx\Gamma_G\left(\overline{\mathrm{Sym}}\,\mathfrak{p}^*\right)$$

If $(G|\mathbf{S}G)$ is split then $\Lambda(\mathbf{S}^*G\oplus\mathbf{S}^*G)$ is also bihomogeneous and $k,p\in[\Lambda(S^*\oplus S^*)\otimes(\mathfrak{g}\oplus S)]^G$ induces an even, nilpotent and biinvariant vector field on $\big(K|\Lambda(\mathbf{S}^*K\oplus\mathbf{S}^*K)\big)$, denoted ∇^L_k. Likewise p induces a smooth supermap

$$(\mathrm{id}|\Delta_p):(K|\Lambda\mathbf{S}^*K)\to\big(K|\Lambda(\mathbf{S}^*K\oplus\mathbf{S}^*K)\big)$$

given by $\alpha_1\wedge\cdots\wedge\alpha_k\mapsto(\alpha_1\circ p)\wedge\cdots\wedge(\alpha_k\circ p)$.

***Theorem** (Multiplication on a split Lie supergroup)* - *The multiplication of a split Lie supergroup is given by*

$$\mathrm{M}^{\text{split}}=\mathrm{twist}\circ m^*\circ\exp(\nabla^L_k)\circ\Delta_p$$

General Lie supergroups

The bundle $\mathcal{R}M$ is not evidently bihomogeneous.

There is a biaction in $\Gamma(\mathcal{R}G)$:

1. By associativity: $(\mathrm{M}\otimes\mathrm{id})\circ\mathrm{M}=(\mathrm{id}\otimes\mathrm{M})\circ\mathrm{M}\colon\Gamma_G(\mathcal{R}G)\to\Gamma_{G\times G\times G}\big(\mathcal{R}G\hat{\boxtimes}\mathcal{R}G\hat{\boxtimes}\mathrm{R}G\big)$.
2. For all $x,y\in G$ we can restrict to fibres: $\mathrm{ev}_{x^{-1}}\otimes\mathrm{id}\otimes\mathrm{ev}_{y^{-1}}\colon\Gamma_{G\times G\times G}\big(\mathcal{R}G\hat{\boxtimes}\mathcal{R}G\hat{\boxtimes}\mathcal{R}G\big)\to\mathcal{R}_{x^{-1}}G\hat{\boxtimes}\Gamma_G(\mathcal{R}G)\hat{\boxtimes}\mathcal{R}_{y^{-1}}G$.
3. Finally $\varepsilon\otimes\mathrm{id}\otimes\varepsilon\colon\mathcal{R}_{x^{-1}}G\hat{\boxtimes}\Gamma_G(\mathcal{R}G)\hat{\boxtimes}\mathcal{R}_{y^{-1}}G\to\mathbb{R}\otimes\Gamma_G(\mathcal{R}G)\otimes\mathbb{R}\cong\Gamma_G(\mathcal{R}G)$.

***Conjecture** (Natural splitting of Lie supergroups)* - *There exists a unique $E\colon\Gamma(\mathcal{R}G)\to\Gamma(\Lambda\mathbf{S}^*G)$, differential operator along id_G such that*

- *$E_*=\mathrm{id}$ on $\Gamma(\mathbf{S}^*G)$,*
- *E intertwines the biaction of G on $\Gamma(\mathcal{R}G)$ with the one on $\Gamma(\Lambda\mathbf{S}^*G)$ thus making $\mathcal{R}G$ a bihomogeneous bundle; and*
- *$E\hat{\boxtimes}E\circ\mathrm{M}\circ E^{-1}\equiv\mathrm{M}^{\text{split}}\mod\Gamma(\Lambda^{\geqslant(1,1)}\mathbf{S}^*(G\times G))$.*

References

[1] Guajardo, Óscar, *A classical approach to smooth supermanifolds*, arxiv:1410.7857.