Singular fibers of integrable systems: a combinatorial point of view

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Remark: Corresponding Hamiltonian flows have the Lax form

$$\frac{\mathrm{d}}{\mathrm{d}t}(X + \lambda B) = [X + \lambda B, \dots] .$$

Fibers and algebraic curves

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Regular fibers

Classical result: Each regular fiber is, modulo conjugation by diagonal matrices, an open subset in the Jacobian of the corresponding curve (the complement of the theta divisor).

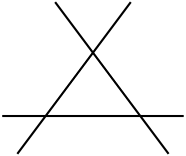
B.A. Dubrovin, P. Van Moerbeke and D. Mumford, A.G. Reyman and M.A. Semenov-Tian-Shansky, M. Adler and P. van Moerbeke.

What about singular fibers?

n straight lines

As the spectral curve, we take n straight lines in general position

$$\prod_{i=1}^{n}(a_i + b_i\lambda - \mu) = 0\,.$$



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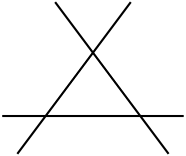

Example:

$$X = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}$$

$$\det(X + \lambda B - \mu \mathrm{Id}) = \prod_{i=1}^{3}(a_i + b_i\lambda - \mu)$$

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Spectral curve:

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$$X = \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \text{ or } \begin{pmatrix} a_1 & 0 \\ * & a_2 \end{pmatrix}$$

Then

$$\det(X + \lambda B - \mu \mathrm{Id}) = (a_1 + b_1\lambda - \mu)(a_2 + b_2\lambda - \mu)\,.$$

n = 2

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And that's it. The fiber corresponding to 2 straight lines is 2 disks intersecting at a point.

n = 3

$$
\begin{array}{ccccc}
 & \begin{pmatrix} a_1 & * & * \\ 0 & a_2 & * \\ 0 & 0 & a_3 \end{pmatrix} & \text{———} & \begin{pmatrix} a_1 & 0 & * \\ * & a_2 & * \\ 0 & 0 & a_3 \end{pmatrix} & \\
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We have a 3-degrees-of-freedom integrable system.

The point

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

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In fact, the fiber corresponding to 3 lines has 7 irreducible components. The seventh component has a double point.

2,7, 38

The number of irreducible components is $2, 7, 38, \ldots$.

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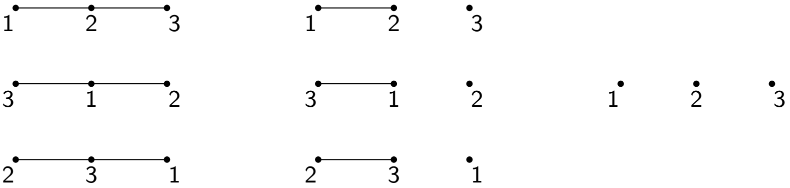
It is sequence A001858 in the online encyclopedia of integer sequences:

$$2, 7, 38, 291, 2932, \ldots$$

1) Number of forests on n labeled vertices.



2) Number of possible score sequences in a round-robin tournament.

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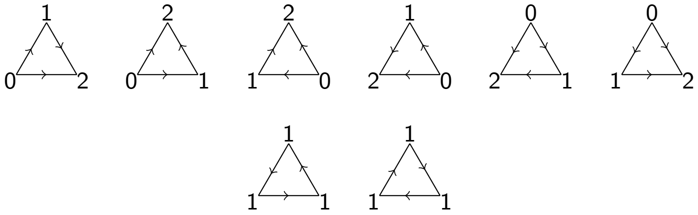

3) Number of integer points in the permutohedron P_n

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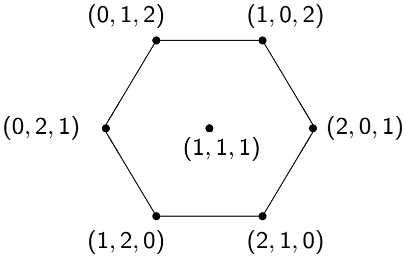

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5) Number of weights for the representation of $\mathfrak{gl}_n(\mathbb{C})$ **with highest weight**

$$\lambda = (0, 1, \ldots, n-1).$$

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Conjecture: For other Lie algebras, one should replace the permutohedron with the corresponding weight polytope.

Thank you for your attention!