Integrable geodesic flows on 2-torus and the systems of hydrodynamical type

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Let

$$ds^2 = \sum_{i,j=1}^{2} g_{ij}(q) dq^i dq^j$$

be a Riemannian metric on the 2-torus T^2. The geodesic flow is called integrable, if the Hamiltonian system

$$\dot{q}^j = \frac{\partial H}{\partial p_j}, \ \dot{p}^j = -\frac{\partial H}{\partial q_j}, \ H = \sum_{i,j=1}^{2} g^{ij}(q) p^i p^j$$

admits a first integral $F : T^*T^2 \to \mathbb{R}$ functionally independent with H almost everywhere

$$\{F, H\} = \sum_{j=1}^{2} \left(\frac{\partial F}{\partial q^j} \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q^j} \frac{\partial F}{\partial p_j} \right) = 0.$$

There are two classically known classes of metrics with integrable geodesic flows. They are written in a conformal coordinates as follows:

$$1.\ ds^2 = f(x)(dx^2 + dy^2), \quad 2.\ ds^2 = (f(x) + g(y))(dx^2 + dy^2)$$

In the first case the Hamiltonian system admits a one parametric group of symmetries and has the integral which is a polynomial of the first degree with respect to momenta. While in the second case the integral appears to be of the second degree and is related to separation of variables in Hamilton- Jacobi equation. The existence of the metrics on T^2 with the integrable geodesic flows having polynomial integrals of the degree higher than 2 and non-reducible to the integrals of the 1 and 2 degree, is not known.

Theorem (Bialy, M.) *Suppose that the Hamiltonian system has an integral F, which is a homogeneous polynomial of degree n. Then on the covering plane $\mathbb{R}^2$ there exist the global coordinates (t, x), where the metric has the following form*

$$ds^2 = g^2(t,x)dt^2 + dx^2,$$

and the integral F can be written in the form

$$F_n = \sum_{k=0}^{n} \frac{a_k(t,x)}{g^{n-k}} p_1^{n-k} p_2^k,$$

Where the last two coefficients can be normalized to be $a_{n-1} \equiv g$ and $a_n \equiv 1$.

Then the commutation relation $\{F, H\} = 0$ *is*

$$U_t + A(U)U_x = 0,$$

where the matrix A *has the form:*

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & a_1 \\ a_{n-1} & 0 & \dots & 0 & 0 & 2a_2 - na_0 \\ 0 & a_{n-1} & \dots & 0 & 0 & 3a_3 - (n-1)a_1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{n-1} & 0 & (n-1)a_{n-1} - 3a_{n-3} \\ 0 & 0 & \dots & 0 & a_{n-1} & na_n - 2a_{n-2} \end{pmatrix},$$

The functions a_i*,* g *are periodic on the variable* x*, and quasi-periodic on the variable* t*.*

Theorem (Bialy, M.) *The system is the semi-Hamiltonian quasi-linear system. More precisely:*

1. In the region of hyperbolicity (all eigenvalues are real and distinct) there exists a change of variables (Riemann invariants) $(a_0, ..., a_{n-1}) \to (r_1, ..., r_n)$ *transforming the system to a diagonal form:*

$$(r_i)_t + \lambda_i(r_1, ..., r_n)(r_i)x = 0, i = 1, .., n.$$

2. There exists a regular change of variables $(a_0, ..., a_{n-1}) \to (G_1, ..., G_n)$ *such that* $G_i, i = 1, .., n$ *are conservation laws:*

$$(G_i(a_0, ..., a_{n-1}))_t + (H_i(a_0, ..., a_{n-1}))_x = 0, i = 1, .., n.$$

Let's fix the energy level of the Hamiltonian $H = \frac{1}{2}(\frac{p_1^2}{g^2} + p_2^2) = \frac{1}{2}$. Assume that $p_1 = g\cos\varphi,\ p_2 = \sin\varphi$, where φ is an angular coordinate in the fibre. Then $F = F(t,x,\varphi)$ becomes a trigonometric polynomial. We have

$$F = F(t,x,\varphi) = \sum_{k=0}^{n} a_k cos^{n-k}\varphi\ sin^k\varphi; \qquad a_{n-1} = g, a_n = 1.$$

$$\frac{dF}{d\tau} = F_t\dot{t} + F_x\dot{x} + F_\varphi\dot{\varphi} = F_t\frac{\cos\varphi}{g} + F_x\sin\varphi + F_\varphi\dot{\varphi} = 0.$$

$$\chi_A(\lambda) = -\frac{g^{n-1}}{\cos^n\varphi}F_\varphi(\varphi),$$

where χ_A is the characteristic polynomial of the matrix A with the relation $\lambda = g\tan\varphi$.

$$(r_i)_t + \lambda_i(r_i)_x = 0,\ r_i = F(\varphi_i),\ i = 1,..,n,$$

Systems of hydrodynamical tape

$$\partial_{r_k}\left(\frac{\partial_{r_i}\lambda_j}{\lambda_i-\lambda_j}\right)=\partial_{r_i}\left(\frac{\partial_{r_k}\lambda_j}{\lambda_k-\lambda_j}\right),$$

$$\Gamma^i_{ij}=\partial_{r_j}\log\sqrt{g_{ii}}=\left(\frac{\partial_{r_j}\lambda_i}{\lambda_j-\lambda_i}\right),$$

$$ds^2=g_{11}(r)(dr_1)^2+\cdots+g_{nn}(dr_n)^2.$$

Generalized hodograph method

$$\frac{\partial_{r_j}w_i}{w_i-w_j}=\frac{\partial_{r_j}\lambda_i}{\lambda_j-\lambda_i},$$

$$w_i=\lambda_i t+x,\ \ i=1,\ldots,n.$$

Theorem (Bialy, M.) *Let $n = 3$, then one has the following alternative:*

Either metric is flat in the region Ω_e or F_3 is reducible on Ω_e , that is it can be written as combination of H and F_1

$$F_3 = k_1 F_1^3 + 2k_2 H F_1$$

for some explicit constants k_1, k_2.

Corollary *We have for the conformal model (c):*

Either metric ρ is flat on Ω_e or $\Lambda = \Lambda(mq_1 + nq_2)$ on Ω_e for some reals m, n; If in addition ρ is known to be real analytic metric on $\mathbb{T}^2$ then $\Lambda = \Lambda(mq_1 + nq_2)$ everywhere on the whole torus $\mathbb{T}^2$ and the flow ρ^t necessarily has a first power integral on the whole torus $\mathbb{T}^2$.

Theorem (Bialy, M.) *Let $n = 4$, then the following alternative holds: Either metric ρ is flat on Ω_e or F_4 is reducible, that is it can be expressed on Ω_e as*

$$F_4 = k_1 F_2^2 + 2k_2 H F_2 + 4k_3 H^2$$

where F_2 is a polynomial of degree 2 which is an integral of the geodesic flow on Ω_e and k_i are constants.

Corollary *The conformal factor $\Lambda(q_1, q_2)$ can be written on Ω_e in the form*

$$\Lambda(q_1, q_2) = f(m_1 q_1 + n_1 q_2) + g(m_2 q_1 + n_2 q_2) \quad with \quad \frac{m_1}{n_1}\frac{m_2}{n_2} = -1.$$

If in addition Λ is known to be real analytic then Λ can be written in such a form for all q_1, q_2 on $\mathbb{T}^2$.

Let $ds^2 = \Lambda(dx^2+dy^2)$ be a metric on T^2. We assume that the geodesic flow has a polynomial in momenta integral

$$F = a_0(x,y)p_1^n + a_1(x,y)p_1^{n-1}p_2 + \cdots + a_n(x,y)p_2^n.$$

Kozlov and Denisova proved that if Λ is trigonometric polynomial then the geodesic flow has no irreducible polynomial integrals of degree higher than two. By Kolokoltsov's theorem

$$a_{n-1} = c_1 + a_{n-3} - a_{n-5} + \dots, \quad a_n = c_2 + a_{n-2} - a_{n-4} + \dots,$$

where c_1, c_2 are constants. The condition $\{H,F\} = 0$, where $H = \frac{p_1^2+p_2^2}{2\Lambda}$ is equivalent to the system of quasi-linear equations

$$A(U)U_x + B(U)U_y = 0, \quad U = (a_0, a_1, \dots, a_{n-2}, \Lambda).$$

This system also can be written in conservation laws form and in the hyperbolic area it has Riemannian invariants, so the system is semi-Hamiltonian.

For semi-Hamiltonian systems

$$\partial_{r_j}\frac{\partial_{r_i}\lambda_k}{\lambda_i-\lambda_k}=\partial_{r_i}\frac{\partial_{r_j}\lambda_k}{\lambda_j-\lambda_k},\qquad i\neq j\neq k.$$

These identities mean that there is a diagonal metric

$$ds^2=H_1^2(r)dr_1^2+\cdots+H_n^2(r)dr_n^2$$

with Christoffel's symbols $\Gamma^k_{ki}=\frac{\partial_{r_i}\lambda_k}{\lambda_i-\lambda_k},\quad i\neq k.$

Theorem (Bialy, M.) *The metric is a metric of Egorov type, i.e. the rotation coefficients* $\beta_{ij}=\frac{\partial_{r_i}H_j}{H_i},\quad i\neq j$ *are symmetric* $\beta_{ij}=\beta_{ji}$, *or equivalently there is a function* $A(r)$ *such that* $\partial_{r_i}A(r)=H_i^2(r)$.

Pavlov, Tsarev:

$$P(U)_x+Q(U)_y=0,\qquad Q(U)_x+R(U)_y=0.$$

Theorem (Bialy, M.) *Let* $n = 3$*, and* $\lambda(x, y)$ *be a periodic solution of the equation*

$$\Delta\lambda = \frac{3c_2}{2}\Lambda - 2a_{11} - 2a_{22}.$$

Then the function λ *satisfies the equation*

$$2\lambda_{xx}\lambda_{xxy} + \lambda_{yy}(\lambda_{xxy} - \lambda_{yyy}) + \lambda_{xy}(\lambda_{xxx} + \lambda_{xyy}) + 4a_{11}\lambda_{xxy} +$$

$$2a_{12}(\lambda_{xxx} + \lambda_{xyy}) + 2a_{22}(\lambda_{xxy} - \lambda_{yyy}) = 0,$$

where a_{11}, a_{12}, a_{22} *are some constants defined by the metric and the integral.*

Theorem (Bialy, M.) *Let $n = 4$, and $\lambda(x, y)$ be a periodic solution of the equation $\Delta\lambda = 2c_2\Lambda - 2a_{11} - 2a_{22}$. Then the function λ satisfies the equation*

$$\lambda_{xy}(\lambda_{yyyy} - \lambda_{xxxx}) + 3(\lambda_{yyy}\lambda_{xyy} - \lambda_{xxy}\lambda_{xxx}) +$$

$$+2(\lambda_{yy}\lambda_{xyyy} - \lambda_{xx}\lambda_{xxxy}) + 4a_{22}\lambda_{xyyy} - 4a_{11}\lambda_{xxxy} + 2a_{12}(\lambda_{yyyy} - \lambda_{xxxx}) = 0,$$

where a_{11}, a_{12}, a_{22} are some constants defined by the metric and the integral.

Theorem (Bialy, M.) *The equation coincides with the Euler-Lagrange equation of the functional*

$$\mathcal{L}(\lambda) = \int \frac{1}{2}\Big(4\lambda_{xy}(a_{22}\lambda_{yy} - a_{11}\lambda_{xx}) + 2a_{12}(\lambda_{yy}^2 - \lambda_{xx}^2) + \lambda_{xy}(\lambda_{yy}^2 - \lambda_{xx}^2)\Big)\, dx\, dy$$

At $a_{11} = a_{22} = \frac{1}{4\varepsilon}, a_{12} = 0$ we have

$$\lambda_{xxxy} - \lambda_{xyyy} = \varepsilon(\lambda_{xy}(\lambda_{yyyy} - \lambda_{xxxx}) + 3(\lambda_{yyy}\lambda_{xyy} - \lambda_{xxy}\lambda_{xxx}) +$$

$$+2(\lambda_{yy}\lambda_{xyyy} - \lambda_{xx}\lambda_{xxxy})),$$

$$\mathcal{L}(\lambda) = \int \frac{1}{2}\lambda_{xy}(\lambda_{yy} - \lambda_{xx})(\frac{1}{\varepsilon} + \Delta\lambda)dx\ dy.$$

The conformal factor of the metric has the form

$$\Lambda = (\frac{\Delta\lambda}{2} + \frac{1}{2\varepsilon}).$$

The equation has formal solution as a formal power series in ε:

$$\lambda(x,y) = \lambda_0(x,y) + \lambda_1(x,y)\varepsilon + \lambda_2(x,y)\varepsilon^2 + \dots,$$

where ε is a small parameter.

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