An Algebraic Geometric Classification of Superintegrable Systems

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(joint work with Jonathan Kress)

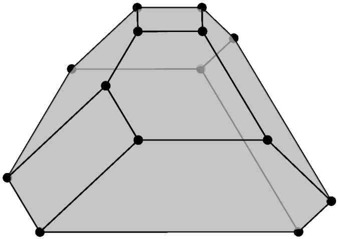
Finite Dimensional Integrable Systems 2015
Będlewo

Classification of separable systems
is an algebraic geometric problem!

Separable systems on S^n . . .

. . . are naturally parametrised by moduli spaces of genus zero stable algebraic curves with marked points.

$$\bar{\mathscr{M}}_{0,n+2}(\mathbb{R})$$



. . . can be classified via Stasheff polytopes

. . . form an operad.

(joint with Alexander P. Veselov)

These objects never appeared before in the theory of separability.

Same idea, different systems!

Classification of superintegrable systems
is an algebraic geometric problem!

Outline

Outline

Superintegrable Systems

Definition

- superintegrable system = $2n-1$ functionally independent integrals

$$K_i + V_i \qquad\qquad i = 1, 2, \ldots, 2n-1$$

- Poisson-commuting with the Hamiltonian

$$g + V$$

- here: K_i quadratic

Superintegrable Systems

Elimination of the potentials V_i

$$\left\{K_i + V_i, g + V\right\} = 0$$

3rd degree: K_i is a 2nd order Killing tensor

1st degree: can be used to

- recover V_i once K_i and V are known

$$dV_i = K_i dV \qquad i = 1, 2, \ldots, 2n-1$$

- eliminate the V_i

$$dK_i dV = 0 \qquad i = 1, 2, \ldots, 2n-1$$

Bertrand-Darboux conditions

Superintegrable Systems

Elimination of the potential V

Dimension $n = 2$:

- The Bertrand-Darboux conditions

$$dK_1 dV = 0 \qquad\qquad dK_2 dV = 0$$

- can be transformed to

$$\begin{bmatrix} V_{zz} \\ V_{\bar{z}\bar{z}} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} V_z \\ V_{\bar{z}} \end{bmatrix} \qquad\qquad \begin{aligned} z &= x + iy \\ \bar{z} &= x - iy \end{aligned}$$

- where the C_{ij} depend rationally on the components of K_1 and K_2 and their derivatives.

SUPERINTEGRABILITY CONDITIONS

AS A SYSTEM OF NONLINEAR PDES

Integrability conditions for

$$\begin{bmatrix} V_{zz} \\ V_{\bar{z}\bar{z}} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} V_z \\ V_{\bar{z}} \end{bmatrix}$$

eliminate the potential V:

$$\begin{aligned} 3C_{21}C_{12,\bar{z}} - C_{11}C_{21,z} - C_{22}C_{12}C_{21} - C_{21}C_{11}C_{11} &= 0 \\ 3C_{12}C_{21,z} - C_{22}C_{12,\bar{z}} - C_{11}C_{21}C_{12} - C_{12}C_{22}C_{22} &= 0 \\ 2C_{12,\bar{z}}C_{21,z} - C_{11}C_{21}C_{12,\bar{z}} - C_{22}C_{12}C_{21,z} & \\ +C_{12}C_{12}C_{21}C_{21} - C_{11}C_{22}C_{12}C_{21} &= 0. \end{aligned}$$

We call these the superintegrability conditions.

Outline

A SIMPLE OBSERVATION

Superintegrability conditions:

$$3C_{21}C_{12,\bar{z}} - C_{11}C_{21,z} - C_{22}C_{12}C_{21} - C_{21}C_{11}C_{11} = 0$$
$$3C_{12}C_{21,z} - C_{22}C_{12,\bar{z}} - C_{11}C_{21}C_{12} - C_{12}C_{22}C_{22} = 0$$
$$2C_{12,\bar{z}}C_{21,z} - C_{11}C_{21}C_{12,\bar{z}} - C_{22}C_{12}C_{21,z} + C_{12}C_{12}C_{21}C_{21} - C_{11}C_{22}C_{12}C_{21} = 0.$$

- C_{ij} rational in the components of K_1, K_2 and their derivatives.
- The superintegrability conditions are algebraic equations in K_1, K_2 and their derivatives.
- K_1, K_2 lie in the finite dimensional space of Killing tensors.
- They depend only on the subspace spanned by g, K_1 and K_2.

A SIMPLE CONSEQUENCE

- The superintegrability conditions define a projective variety.
- a subvariety in the Grassmannian of $(2n-1)$-dimensional subspaces in the space of Killing tensors.
- equipped with a natural group action of the isometry group G.

A NEW POINT OF VIEW
ON SUPERINTEGRABLE SYSTEMS

That is, the classification problem . . .

- has been solved in the category of sets.
- but should be solved in its natural category:

In the category of projective G-varieties.

This will simplify and enrich the classification!

ALGEBRAIC GEOMETRIC DESCIPTION
OF SUPERINTEGRABLE SYSTEMS

Dimension $n = 2$:

- The space of Killing tensors is isomorphic to $\mathbb{R}^6$.
- A superintegrable system is an element of the Grassmannian
$$G_3(\mathbb{R}^6).$$
- Factoring out the metric reduces this to
$$G_2(\mathbb{R}^5).$$
- Plücker embedding:
$$\begin{array}{ccc} G_2(\mathbb{R}^5) & \hookrightarrow & \mathbb{P}(\Lambda^2\mathbb{R}^5) \\ \langle K_1, K_2 \rangle & \mapsto & K_1 \wedge K_2 \end{array}$$

CONCLUSION

Every superintegrable system in dimension two is encoded in a rank two skew symmetric 5×5 matrix.

ALGEBRAIC DESCRIPTION

OF SUPERINTEGRABLE SYSTEMS

$$\begin{bmatrix} 0 & a_{30} & a_{20} & a_{10} & a_{00} \\ -a_{30} & 0 & a_{21} & a_{11} & a_{01} \\ -a_{20} & -a_{21} & 0 & a_{12} & a_{02} \\ -a_{10} & -a_{11} & -a_{12} & 0 & a_{03} \\ -a_{00} & -a_{01} & -a_{02} & -a_{03} & 0 \end{bmatrix}$$

Define a ternary cubic

$$D(z,\bar{z}) = a_{21}z^2\bar{z} + a_{12}z\bar{z}^2 + a_{20}z^2 + a_{02}\bar{z}^2 + a_{11}z\bar{z} + a_{10}z + a_{01}\bar{z} + a_{00}$$

and two quadratic polynomials

$$A(\bar{z}) = a_{21}\bar{z}^2 + a_{20}\bar{z} + a_{30} \qquad B(z) = a_{12}z^2 + a_{02}z + a_{03}$$

then:

$$\begin{bmatrix} V_{zz} \\ V_{\bar{z}\bar{z}} \end{bmatrix} = \frac{1}{D} \begin{bmatrix} -D_z & A \\ B & -D_{\bar{z}} \end{bmatrix} \begin{bmatrix} V_z \\ V_{\bar{z}} \end{bmatrix}$$

ALGEBRAIC SUPERINTEGRABILITY CONDITIONS

Integrability conditions for

$$\begin{bmatrix} V_{zz} \\ V_{\bar{z}\bar{z}} \end{bmatrix} = \frac{1}{D} \begin{bmatrix} -D_z & A \\ B & -D_{\bar{z}} \end{bmatrix} \begin{bmatrix} V_z \\ V_{\bar{z}} \end{bmatrix}$$

yield algebraic superintegrability conditions:

$$\begin{aligned} D_z^2 D_{\bar{z}\bar{z}} + D_{\bar{z}}^2 D_{zz} + D_z D_{\bar{z}} D_{z\bar{z}} \\ = D(2D_{zz} D_{\bar{z}\bar{z}} + D_z D_{\bar{z}\bar{z}z} + D_{\bar{z}} D_{zz\bar{z}} + D_{z\bar{z}}^2) \\ AD_{\bar{z}}^2 = 2DD_{\bar{z}} D_{zz} - \tfrac{3}{2} D^2 D_{zz\bar{z}} - D_{\bar{z}} D_z^2 + DD_z D_{z\bar{z}} \\ BD_z^2 = 2DD_z D_{\bar{z}\bar{z}} - \tfrac{3}{2} D^2 D_{\bar{z}\bar{z}z} - D_z D_{\bar{z}}^2 + DD_{\bar{z}} D_{z\bar{z}}. \end{aligned}$$

REMARK

$D(z, \bar{z})$ completely determines the superintegrable system (except for the degenerated cases where $D_z = 0$ or $D_{\bar{z}} = 0$).

Outline

The Key Observation

Fact

The algebraic superintegrability conditions imply that the ternary cubic $D(z, \bar{z})$ decomposes into linear factors:

$$D(z, \bar{z}) = (a_1 z + b_1 \bar{z} + c_1)(a_2 z + b_2 \bar{z} + c_2)(a_3 z + b_3 \bar{z} + c_3)$$

Proof.

Derivation of the first algebraic superintegrability condition with respect to z and $\bar{z}$ yields:

$$2 D_{zz\bar{z}} D_{\bar{z}\bar{z}z} D = D_{zz} D_{z\bar{z}} D_{\bar{z}\bar{z}}$$

i.e.

$$2 \cdot \text{constant} \cdot \text{constant} \cdot D = \text{linear} \cdot \text{linear} \cdot \text{linear}$$

□

GEOMETRIC INTERPRETATION

OF THE DECOMPOSITION OF $D(z, \bar{z})$ INTO LINEAR FACTORS

$$D(z, \bar{z}) = (a_1 z + b_1 \bar{z} + c_1)(a_2 z + b_2 \bar{z} + c_2)(a_3 z + b_3 \bar{z} + c_3)$$

Remember:

$$\begin{bmatrix} V_{zz} \\ V_{\bar{z}\bar{z}} \end{bmatrix} = \frac{1}{D} \begin{bmatrix} -D_z & A \\ B & -D_{\bar{z}} \end{bmatrix} \begin{bmatrix} V_z \\ V_{\bar{z}} \end{bmatrix}$$

COROLLARY

- *The singular set of a superintegrable potential is an arrangement of three complex projective lines (counted with multiplicities).*
- *This singular set (almost) completely determines the superintegrable potential.*

SOLUTION

OF THE ALGEBRAIC SUPERINTEGRABILITY CONDITIONS

$$D(z,\bar{z}) = (a_1 z + b_1\bar{z} + c_1)(a_2 z + b_2\bar{z} + c_2)(a_3 z + b_3\bar{z} + c_3)$$

substituted into the algebraic superintegrability conditions yields:

$D(z,\bar{z})$	condition
$(a_1 z + c_1)(a_2 z + b_2\bar{z} + c_2)(b_3\bar{z} + c_3)$	$\det\begin{bmatrix} a_1 & 0 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{bmatrix} = 0$
$(a_1 z + c_1)(a_2 z + c_2)(b_3\bar{z} + c_3)$	
$(a_1 z + c_1)(b_2\bar{z} + c_2)(b_3\bar{z} + c_3)$	
$(a_1 z + b_1\bar{z} + c_1)(a_3 z + b_3\bar{z} + c_3)$	$\det\begin{bmatrix} a_1 & -b_1 & 0 \\ 0 & 0 & 1 \\ a_3 & b_3 & 0 \end{bmatrix} = 0$

The variety of superintegrable systems

Components

- determinantal variety

$$\det \begin{bmatrix} a_1 & 0 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{bmatrix} = 0$$

- two conjugated $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
- determinantal variety

$$\det \begin{bmatrix} a_1 & -b_1 & 0 \\ 0 & 0 & 1 \\ a_3 & b_3 & 0 \end{bmatrix} = 0$$

THE VARIETY OF SUPERINTEGRABLE SYSTEMS

ISOMETRY GROUP ACTION

- isometry group action = induced action on polynomials

$$D(z,\bar{z}) \qquad A(\bar{z}) \qquad B(z)$$

- rotations = shears

$$z \mapsto \lambda z \qquad z \mapsto z$$
$$\bar{z} \mapsto \bar{z} \qquad \bar{z} \mapsto \bar{z}/\lambda$$

- translations = shifts

$$z \mapsto z + c \qquad z \mapsto z$$
$$\bar{z} \mapsto \bar{z} \qquad \bar{z} \mapsto \bar{z} + \bar{c}$$

- The isometry group action is linear!

Superintegrable systems

normal forms

new	$D(z, \bar{z})$	$A(\bar{z})$	$B(z)$	old
$(1, 1, 1)$	$z(z + \bar{z})\bar{z}$	$\bar{z}^2$	z^2	E16
$(11, 0, 1)$	$z(z + 1)\bar{z}$	$\bar{z}^2$	0	E19
$(2, 0, 1)$	$z^2\bar{z}$	$\bar{z}^2$	0	E17
$(0, 11, 0)$	$(z + \bar{z})(z - \bar{z})$	$2\bar{z}$	$-2z$	E1
$(11, 0, 0)$	$z(z + 1)$	$2\bar{z}$	0	E7
$(2, 0, 0)$	z^2	$2\bar{z}$	0	E8
$(1, 0, 1)$	$z\bar{z}$	0	0	E20
$(0, 1, 0)$	$z + \bar{z}$	1	1	E2
$(1, 0, 0)$	z	$0/1$	0	E11 / E9
$(0, 0, 0)$	1	$0/1$	0	E3 / E10

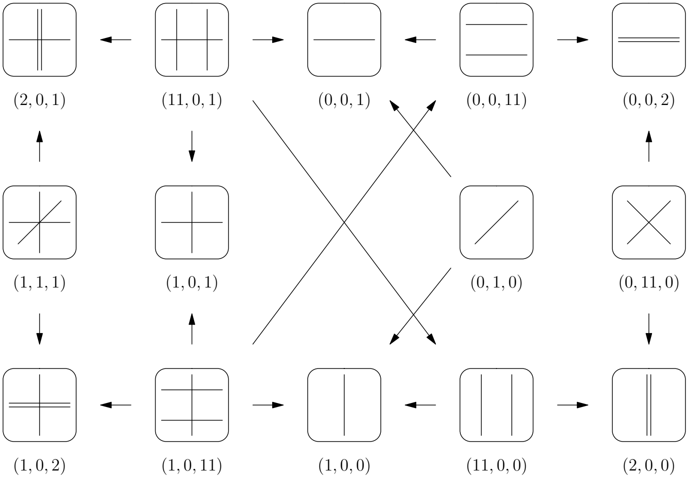
(plus conjugated forms)

SUPERINTEGRABLE POTENTIALS

$(1,1,1)$	$\frac{1}{\sqrt{x^2+y^2}}\left(\alpha+\frac{\beta}{x+\sqrt{x^2+y^2}}+\frac{\gamma}{x-\sqrt{x^2+y^2}}\right)$
$(11,0,1)$	$\frac{\alpha\bar{z}}{\sqrt{\bar{z}^2-1}}+\frac{\beta}{\sqrt{z(\bar{z}+1)}}+\frac{\gamma}{\sqrt{z(\bar{z}-1)}}$
$(2,0,1)$	$\frac{\alpha}{\sqrt{z\bar{z}}}+\frac{\beta}{\bar{z}\sqrt{z\bar{z}}}+\frac{\gamma}{\bar{z}^2}$
$(0,11,0)$	$\alpha(x^2+y^2)+\frac{\beta}{x^2}+\frac{\gamma}{y^2}$
$(11,0,0)$	$\alpha z\bar{z}+\beta\frac{\bar{z}}{\sqrt{\bar{z}^2-1}}+\gamma\frac{z(2\bar{z}^2-1)}{\sqrt{\bar{z}^2-1}}$
$(2,0,0)$	$\alpha z\bar{z}+\frac{\beta}{\bar{z}^2}+\frac{\gamma z}{\bar{z}^3}$
$(1,0,1)$	$\frac{\alpha}{\sqrt{z\bar{z}}}+\frac{\beta}{\sqrt{z}}+\frac{\gamma}{\sqrt{\bar{z}}}$
$(0,1,0)$	$\alpha(x^2+4y^2)+\frac{\beta}{x^2}+\gamma y$
$(1,0,0)$	$\frac{\alpha z}{\sqrt{\bar{z}}}+\frac{\beta}{\sqrt{\bar{z}}}+\gamma z$ / $\frac{\alpha}{\sqrt{\bar{z}}}+\beta(z+\bar{z})+\gamma\frac{z+3\bar{z}}{\sqrt{\bar{z}}}$
$(0,0,0)$	$\alpha z\bar{z}+\beta z+\gamma\bar{z}$ / $\alpha\bar{z}\left(\bar{z}^2+3z\right)+\beta\left(\bar{z}^2+z\right)+\gamma\bar{z}$

Superintegrable systems

degenerations



ADVANTAGES

OF THE ALGEBRAIC GEOMETRIC APPROACH

Why solve a solved problem?

- simpler
- more geometric
- more structure
- linear isometry group action
- limits become curves on a variety
- generalisation to higher dimensions

IN THE NEXT EPISODE . . .

FDIS 2017 BARCELONA

An algebraic geometric classification of superintegrable systems . . .

- sphere
- degenerated
- conformal
- dimension three
- arbitrary dimension
- higher order

(joint with Andreas Vollmer)

In the next episode . . .

FDIS 2017 Barcelona

An algebraic geometric interpretation of . . .

- İnönü-Wigner contractions
- Stäckel transforms
- coupling constant metamorphosis
- Askey scheme of hyperbolic orthogonal polynomials
- more general families of special functions

Programme

Algebro-geometrisation of the classification of superintegrable systems and its applications.

In the next episode . . .

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Your example here.

Thanks for your attention!