Periodic Orbits in the Kepler-Heisenberg Problem

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LEAP MOTION

Abstract

The Kepler problem is among the oldest and most fundamental problems in mechanics. It has been studied in curved spaces, such as the sphere and hyperbolic plane. Here, we formulate the problem on the Heisenberg group, the simplest sub-Riemannian manifold. We take the sub-Riemannian Hamiltonian as our kinetic energy, and our potential is the fundamental solution to the Heisenberg sub-Laplacian. The resulting dynamical system is known to contain a fundamental integrable subsystem. We discuss the use of variational methods in proving the existence of periodic orbits with k-fold rotational symmetry for any odd integer k greater than one, and show approximations for k=3, 5, 7. Numerical methods which take advantage of the variational formulation are used to find approximate solutions having the sought-after symmetries. The sub-Riemannian structure on the Heisenberg group allows us to parameterize the optimization problem in terms of a single complex-valued curve, the Fourier decomposition of which lends itself to a particularly simple expression of the symmetry conditions.

Background

In [2] we introduced the Kepler-Heisenberg problem and recorded many surprising properties. The object of interest is a dynamical system which is intended to model the motion of a planet around a sun if the ambient geometry were the three dimensional Heisenberg group equipped with its sub-Riemannian structure.

In Hamiltonian mechanics, one typically begins with a Riemannian manifold and a choice of potential energy function. The Riemannian metric induces a kinetic energy function on the cotangent bundle of the manifold. On the Heisenberg group, we have a natural choice of kinetic energy, induced by the sub-Riemannian metric, which indeed generates the sub-Riemannian geodesics. We choose as our potential the fundamental solution to the Heisenberg sub-Laplacian, given explicitly by Folland in [1]. The delta function source, acting as our sun, lies at the origin. This characterization of gravitational potential is not original, and is guided by the fact that 1/4πr is the fundamental solution to the Laplacian on R^3.

Newton studied the Euclidean Kepler Problem in the 17th century and derived Kepler's three laws of planetary motion. But the problem was posed on spaces of constant curvature much later. In 1835, Lobachevsky posed the Kepler Problem in three-dimensional hyperbolic space. Bolyai did similar work in the same time period. Paul Joseph Serret posed and solved the Kepler Problem on the two-sphere in 1860. Schering, Lipschitz, Killing, and Liebmann studied the Kepler Problem on hyperbolic and spherical three-space between 1870 and 1902. With this historical background in mind, it seems natural to continue efforts to pose and solve the Kepler Problem in more general geometries.

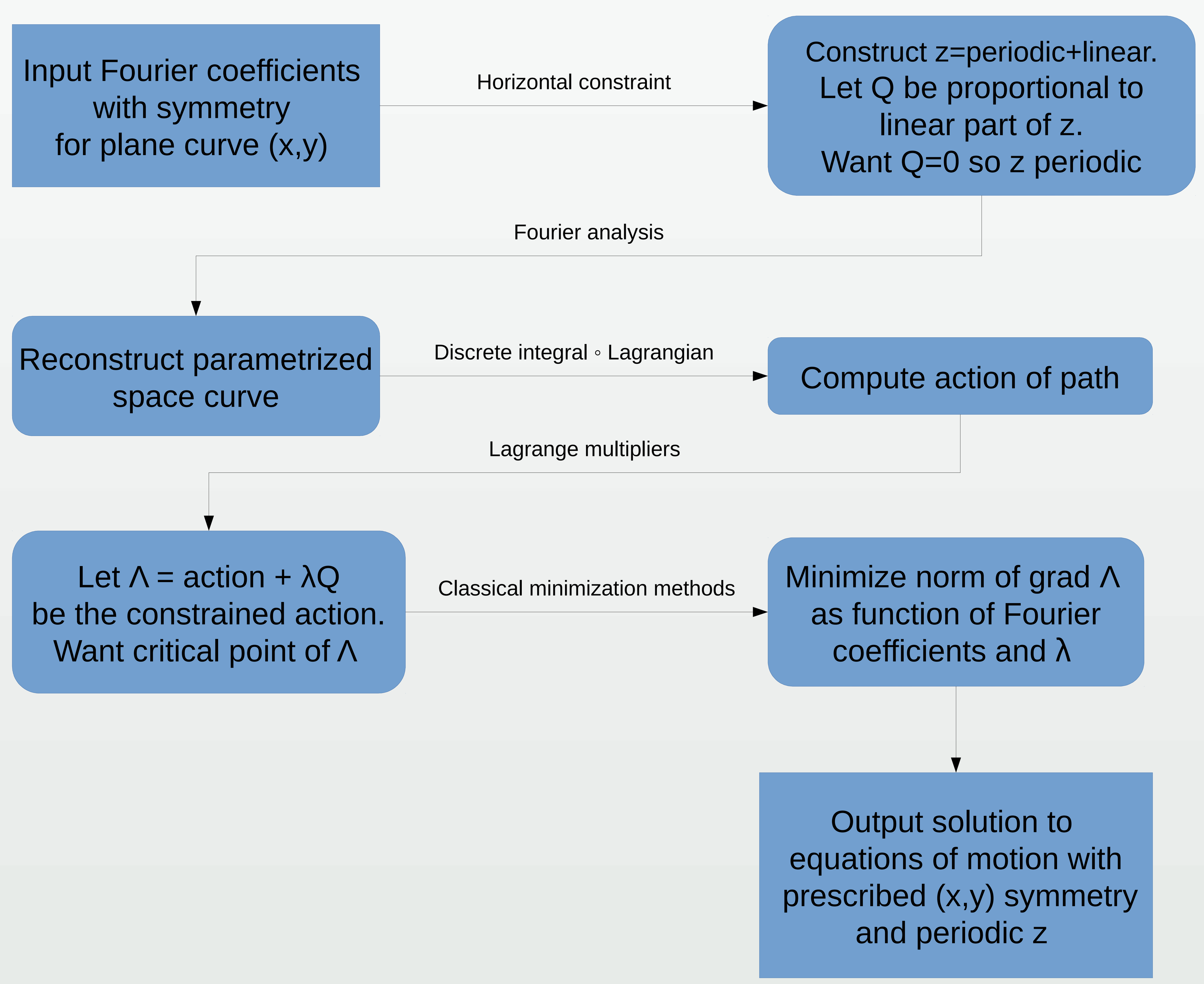
We proved in[2] that phase space for the Kepler-Heisenberg problem contains a fundamental invariant hypersurface on which the dynamics are integrable. In addition, we reduced the integration of this integrable subsystem to the parametrization of a family of algebraic plane curves. We showed further that periodic orbits, should they exist, must lie on this hypersurface. In [4], we proved the existence of periodic orbits with k-fold rotational symmetry for any odd integer k>1. We employed the direct method from the calculus of variations.

Methods

Our Hamiltonian is

$$H = \underbrace{\tfrac{1}{2}(p_x - \tfrac{1}{2}yp_z)^2 + \tfrac{1}{2}(p_y + \tfrac{1}{2}xp_z)^2}_{\text{Kinetic}} - \underbrace{\tfrac{2}{\pi}\left((x^2+y^2)^2 + 16z^2\right)^{-\frac{1}{2}}}_{\text{Potential}}$$

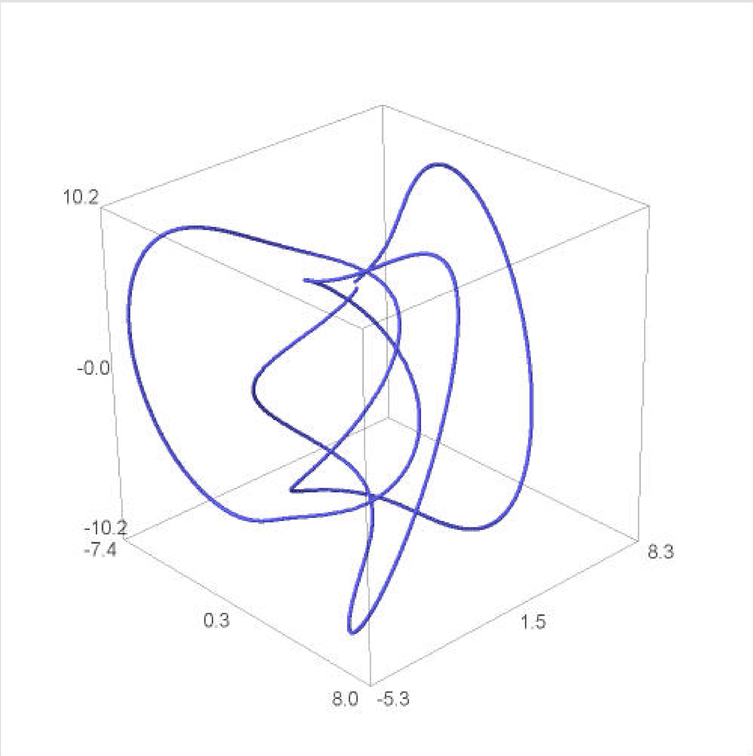
The following algorithm, inspired by [3], finds numerical approximations to solutions of the system with prescribed rotational symmetry.
Here, all variables are functions of time, which is sampled in the execution.

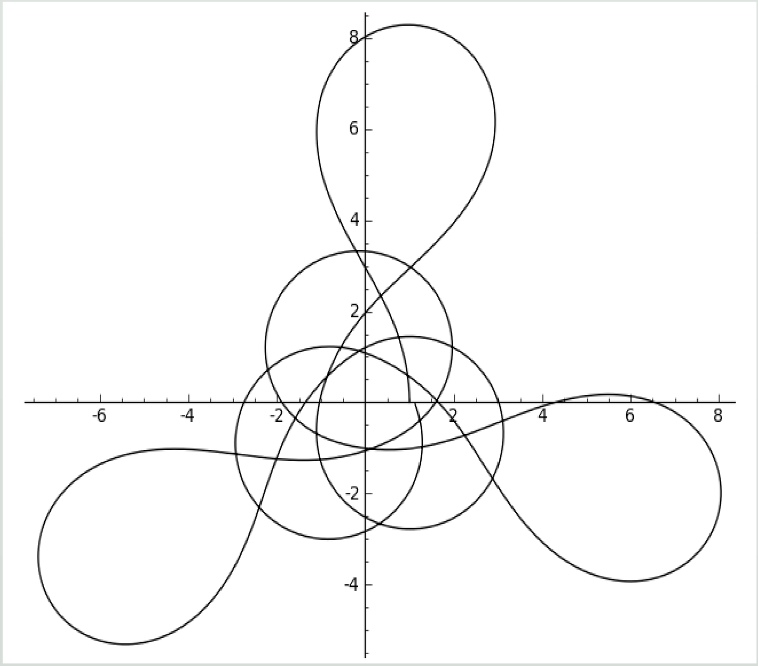


Old Results

Theorem ([4]): For any odd k>1, there exists a periodic orbit with k-fold rotational symmetry about the z-axis.

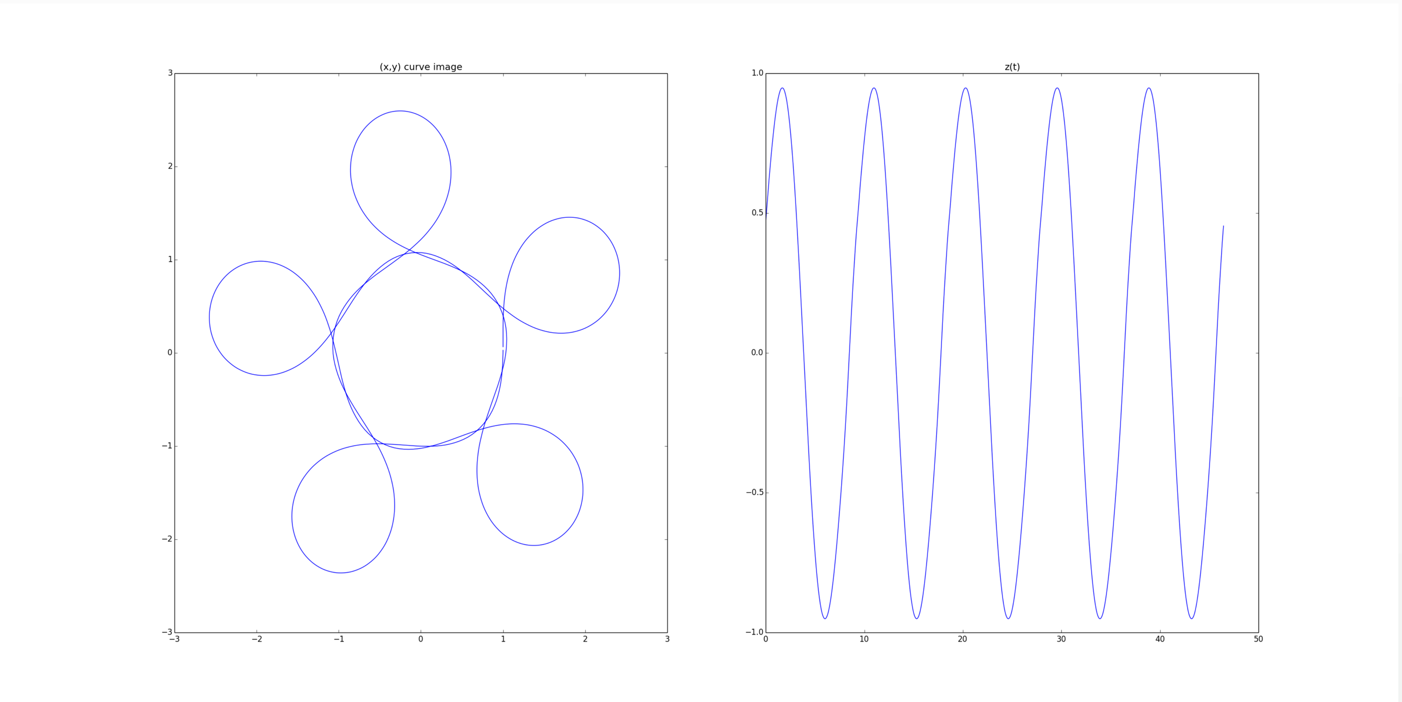
The orbit below enjoys near 3-fold rotational symmetry. It was found by an RK4 integrator and good luck. This inspired the work in [4]. Here we see the orbit in 3-space as well as its projection to the xy-plane. Note that the orbit does not quite close.

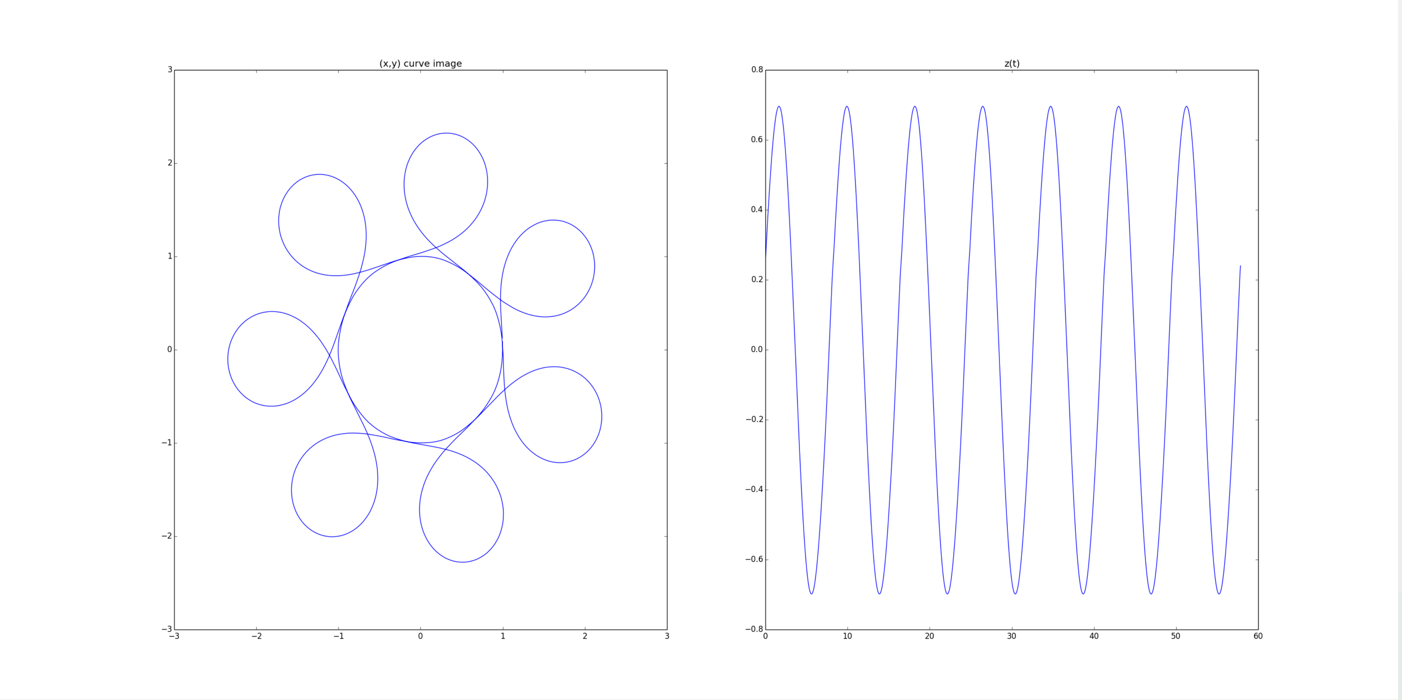




New Results

The orbits below enjoy 5-fold and 7-fold rotational symmetry about the z-axis. Projections to the xy-plane are shown alongside z as a function of time. These were found using the numerical techniques described in Methods.





References

[1] Folland, G. A fundamental solution to a subelliptic operator, Bulletin of the AMS, Vol. 79, 1973

[2] Montgomery, R. and Shanbrom, C. *Keplerian motion on the Heisenberg group and elsewhere*, Geometric Mechanics: The Legacy of Jerry Marsden, Fields Institute Communications Series, 2015

[3] Nauenberg, M. *Periodic orbits for three particles with finite angular momentum*, Phys. Lett. Vol. 292, pp93-99, 2001

[4] Shanbrom, C. *Periodic orbits in the Kepler-Heisenberg problem*, Journal of Geometric Mechanics, Vol. 6, No. 2, pp261-278, 2014

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