Integrability of certain homogeneous Hamiltonian systems

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Abstract

We investigate a class of natural Hamiltonian systems with two degrees of freedom. The kinetic energy depends on coordinates but the system is homogeneous. Thanks to this property it admits, in general case, a particular solution. Using this solution we derive necessary conditions for the integrability of these system investigating differential Galois group of variational equations.

1 Introduction. Morales–Ramis Theory

⋆ Let $H : \mathbb{C}^{2n} \to \mathbb{C}$ be a holomorphic Hamiltonian, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{v}_H(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{C}^{2n}, \quad t \in \mathbb{C}, \tag{1}$$

the associated Hamilton equations.

⋆ Let $t \to \boldsymbol{\varphi}(t) \in \mathbb{C}^{2n}$ be a solution of (1). The maximal analytic continuation of $\boldsymbol{\varphi}(t)$ defines a Riemann surface $\boldsymbol{\Gamma}$ with t as a local coordinate.

⋆ The **variational equations** (VE) along $\boldsymbol{\varphi}(t)$ have the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi} = \boldsymbol{A}(t)\boldsymbol{\xi}, \quad \text{where} \quad \boldsymbol{A}(t) = \frac{\partial \boldsymbol{v}_H}{\partial \boldsymbol{x}}(\boldsymbol{\varphi}(t)). \tag{2}$$

⋆ The variational equations divides into two subsystems: the tangential variational equations of dimension 2 and the **normal variational equations** (NVE) of dimension $2n-2$.

⋆ The **monodromy group** $\mathbb{M}$ of NVE is the image of the fundamental group of $\boldsymbol{\Gamma}$ obtained in the process of continuation of local solutions of NVE along closed paths on $\boldsymbol{\Gamma}$. It is a subgroup of $\mathrm{SL}(2n-2,\mathbb{C})$.

⋆ The **differential Galois group** $\mathcal{G}$ of NVE is a matrix group which acts on solutions of NVE and does not change polynomial relations among them. It is an algebraic subgroup of $\mathrm{SL}(2n-2,\mathbb{C})$. Thus, is a union of disjoint connected components. One of them, containing the identity, is called the identity component of $\mathcal{G}$ and is denoted by $\mathcal{G}^0$. In nineties of XX century Morales-Ruiz and Ramis showed that the integrability in the Liouville sense imposes a very restrictive condition on $\mathcal{G}^0$.

Theorem 1 (Morales-Ruiz and Ramis [1]). *Assume that a Hamiltonian system* (1) *is meromorphically integrable in the Liouville sense in a neighbourhood of a phase curve* $\boldsymbol{\Gamma}$*. Then, the identity component* $\mathcal{G}^0$ *of the differential Galois group* $\mathcal{G}$ *of* VEs *associated with* $\boldsymbol{\Gamma}$ *is Abelian.*

Aplications of Morales–Ramis theory

⋆ to prove non-integrability of Hamiltonian systems,

⋆ to detection possible integrable cases for Hamiltonian systems depending on parameters.

Main steps during applications

⋆ find a particular solution different from equilibrium points,

⋆ calculate VE and NVE,

⋆ check if $\mathcal{G}^0$ is Abelian. This step is the most difficult. Usually we try to transform NVE, by means of transformations which do not change $\mathcal{G}^0$, to equation with known differential Galois group: e.g. the Riemann equation, the Lamé equation, an equation of the second order with rational coefficients. For the differential equations of the second order with rational coefficients there is an algorithm (now called the **Kovacic algorithm** [2]) which allows to decide if an equation of this type has solutions in closed form, to calculate its explicit forms and to relate a form of the solutions with the differential Galois group.

Theorem 1 has found a very effective application for natural system given by the following Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\boldsymbol{q}), \tag{3}$$

where $V(\boldsymbol{q})$ is a homogeneous function of degree $k \in \mathbb{Z}$, and $\boldsymbol{q} = (q_1,\ldots,q_n)$ and $\boldsymbol{p} = (p_1,\ldots,p_n)$ are the generalised coordinates and momenta, respectively. For such systems the particular solutions are know and analysis of differential Galois group of variational equations can be done effectively. The obtained necessary conditions for the integrability have the form of arithmetic restrictions of eigenvalues of Hessian $V''(\boldsymbol{d})$ where $\boldsymbol{d}$ is a nonzero solution of $V'(\boldsymbol{d}) = \boldsymbol{d}$. For details consult e.g. [1].

2 Presentation of the system

Hamiltonian (3) describes a particle moving under influence of potential forces in flat Euclidean space $\mathbb{R}^n$. It is a natural to ask what is an analog of homogeneous systems in curved spaces. There is no obvious answer to this question. We have to take into account the form of metric of the configuration space as well as the form of the potential. Here we consider systems with two degrees of freedom given be the following Hamiltonian

$$H = T + V, \qquad T = \frac{1}{2}r^n\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right), \qquad V = r^m U(\varphi). \tag{4}$$

If we consider (r,φ) as the polar coordinates, then the kinetic energy corresponds to a singular metric on a plane or sphere.

3 Main result

In order to formulate our main result that are necessary conditions for the integrability of Hamiltonian systems given by (4) we need to define the following sets

$$\mathcal{I}_0(k,m) := \left\{\frac{1}{k}(mp+1)(2mp+k) \;|\; p \in \mathbb{Z}\right\}, \tag{5}$$

$$\mathcal{I}_1(k,m) := \left\{\frac{1}{2k}(mp-2)(mp-k) \;|\; p = 2r+1, r \in \mathbb{Z}\right\}, \tag{6}$$

$$\mathcal{I}_2(k,m) := \left\{\frac{1}{8k}\left[4m^2\left(p+\frac{1}{2}\right)^2 - (k-2)^2\right] \;|\; p \in \mathbb{Z}\right\}, \tag{7}$$

$$\mathcal{I}_3(k,m) := \left\{\frac{1}{8k}\left[4m^2\left(p+\frac{1}{3}\right)^2 - (k-2)^2\right] \;|\; p \in \mathbb{Z}\right\}, \tag{8}$$

$$\mathcal{I}_4(k,m) := \left\{\frac{1}{8k}\left[4m^2\left(p+\frac{1}{4}\right)^2 - (k-2)^2\right] \;|\; p \in \mathbb{Z}\right\}, \tag{9}$$

$$\mathcal{I}_5(k,m) := \left\{\frac{1}{8k}\left[4m^2\left(p+\frac{1}{5}\right)^2 - (k-2)^2\right] \;|\; p \in \mathbb{Z}\right\}, \tag{10}$$

$$\mathcal{I}_6(k,m) := \left\{\frac{1}{8k}\left[4m^2\left(p+\frac{2}{5}\right)^2 - (k-2)^2\right] \;|\; p \in \mathbb{Z}\right\}, \tag{11}$$

and $\mathcal{I}_{\mathrm{a}}(k,m) := \mathcal{I}_0(k,m) \cup \mathcal{I}_1(k,m) \cup \mathcal{I}_2(k,m)$.

Theorem 2. *Assume that* $U(\varphi)$ *is a complex meromorphic function and that there exists* $\varphi_0 \in \mathbb{C}$ *such that* $U'(\varphi_0) = 0$ *and* $U(\varphi_0) \neq 0$*. If the Hamiltonian system defined by Hamiltonian* (4) *is integrable in the Liouville sense, then number*

$$\lambda := 1 + \frac{U''(\varphi_0)}{kU(\varphi_0)}, \qquad k = m - n, \tag{12}$$

belongs to set $\mathcal{I}(k,m)$ *which is defined by the following table*

No.	k	m	$\mathcal{I}(k,m)$
1	$-2(pm+1)$	m	$\mathbb{R}$
2	k	m	$\mathcal{I}_{\mathrm{a}}(k,m)$
3	$k = 2(mp-1) \pm \frac{1}{3}m$	$3q$	$\bigcup_{i=0}^{6}\mathcal{I}_i(k,m)$
4	$k = 2(mp-1) \pm \frac{1}{2}m$	$2q$	$\mathcal{I}_{\mathrm{a}}(k,m) \cup \mathcal{I}_4(k,m)$
5	$k = 2(mp-1) \pm \frac{3}{5}m$	$5q$	$\mathcal{I}_{\mathrm{a}}(k,m) \cup \mathcal{I}_3(k,m) \cup \mathcal{I}_6(k,m)$
6	$k = 2(mp-1) \pm \frac{1}{5}m$	$5q$	$\mathcal{I}_{\mathrm{a}}(k,m) \cup \mathcal{I}_3(k,m) \cup \mathcal{I}_5(k,m)$

Table 1: Integrability table. Here $k, m, p, q \in \mathbb{Z}$

4 Proof of Theorem 2

If $U'(\varphi_0) = 0$ for a certain $\varphi_0 \in \mathbb{C}$, then the equations of motion corresponding to Hamiltonian (4) have two dimensional invariant manifold

$$\mathcal{N} = \left\{(r,\varphi,p_r,p_\varphi) \in \mathbb{C}^4 | \varphi = \varphi_0,\; p_\varphi = 0\right\},$$

and its restriction to $\mathcal{N}$ is the following

$$\dot{r} = r^n p_r, \qquad \dot{p}_r = -\frac{1}{2}nr^{n-1}p_r^2 - mr^{m-1}U(\varphi_0). \tag{13}$$

Let $[R,\Phi,P_R,P_\Phi]^T$ denote the variations of $[r,\varphi,p_r,p_\varphi]^T$. Then the NVE after the change of independent variable $t \longrightarrow z = r^m(t)$, for the chosen energy level $E = U(\varphi_0)$ transforms into

$$z(z-1)\Phi''(z) + \left[\frac{2m+k+2}{2m}z - \frac{k+m+2}{2m}\right]\Phi'(z) + \frac{k(1-\lambda)}{2m^2}\Phi(z) = 0, \tag{14}$$

where $k \equiv m-n$, and $\lambda \equiv 1 + U''(\varphi_0)/(kU(\varphi_0))$. We recognize that the equation (14) is the so-called **Gauss hypergeometric differential equation**, see e.g. [1]. In our case the differences of exponents at $z=0$, $z=1$ and at $z=\infty$ are given by

$$\rho = \frac{m-k-2}{2m}, \quad \sigma = \frac{1}{2}, \quad \tau = \frac{\sqrt{(k-2)^2 + 8k\lambda}}{2m}.$$

If Hamilton equations of motion are integrable in the Liouville sense, then by Theorem 1 the identity component of the differential Galois group of variational equations as well as normal variational equations (14) is Abelian, so in particular it is solvable. Necessary and sufficient conditions for solvability of the identity component of the differential Galois group for the Riemann P equation as well as it special form: the hypergeometric equation are well known thanks to the **Kimura theorem** [3]. The proof consists of a direct application of this theorem to our equation (14).

⋆ **Detailed proof can be found in [4]** .

5 Application of Theorem 2

Let us consider Hamiltonian (4) with the specified function $U(\varphi) = -\cos\varphi$. Then $U' = \sin\varphi$ and we take $\varphi_0 = 0$. Since $U''(0)/U(0) = -1$, thus $\lambda = (k-1)/k$. Comparing this value with forms of λ in families $\mathcal{I}_j(k,m)$ for $j = 0,\ldots,6$ we obtain the following admissible values of $m,n \in \mathbb{Z}$ for that system satisfies necessary integrability conditions

$$\begin{array}{llllll} 1. & m=1, & n=6, & 5. & m=2, & n=1,\\ 2. & m=-1, & n=-2, & 6. & m=-2, & n=-3,\\ 3. & m=1, & n=0, & 7. & m=2, & n=7,\\ 4. & m=-1, & n=4, & 8. & m=-2, & n=3. \end{array} \tag{15}$$

Surprisingly cases 1–4 in (15) are maximally superintegrable.

⋆ **Case 1.** In this case we have the Hamiltonian of the following form

$$H = \frac{1}{2}r^6\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r\cos\varphi. \tag{16}$$

This system has two additional, functionally independent first integrals of the second order in momenta

$$\begin{aligned} F_1 &:= r^2p_\varphi^2\cos(2\varphi) - r^3p_rp_\varphi\sin(2\varphi) + r^{-1}\sin\varphi\sin(2\varphi),\\ F_2 &:= r^2p_\varphi^2\sin(2\varphi) + r^3p_rp_\varphi\cos(2\varphi) - r^{-1}\sin\varphi\cos(2\varphi). \end{aligned} \tag{17}$$

Since we have $2n-1 = 3$ integrals of motion H, F_1, F_2 such that $\{F_1,H\} = \{F_2,H\} = 0$ and $\{F_1,F_2\} \neq 0$ the system is maximally super integrable.

⋆ **Case 2.** We have the following Hamiltonian

$$H = \frac{1}{2}r^{-2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r^{-1}\cos\varphi, \tag{18}$$

and two additional functionally independent first integrals are

$$\begin{aligned} F_1 &:= r^{-2}p_\varphi^2\cos(2\varphi) + r^{-1}p_rp_\varphi\sin(2\varphi) + r\sin\varphi\sin(2\varphi),\\ F_2 &:= -r^{-2}p_\varphi^2\sin(2\varphi) + r^{-1}p_rp_\varphi\cos(2\varphi) + r\sin\varphi\cos(2\varphi). \end{aligned} \tag{19}$$

⋆ **Case 3.** Hamilton and additional first integrals are the following

$$\begin{aligned} H &= \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r\cos\varphi,\\ F_1 &:= r^{-1}p_\varphi^2\cos\varphi + p_rp_\varphi\sin\varphi + \frac{1}{2}r^2\sin^2\varphi,\\ F_2 &:= \left(p_r^2 - r^{-2}p_\varphi^2\right)\cos\varphi\sin\varphi + r^{-1}p_rp_\varphi\cos(2\varphi) - r\sin\varphi. \end{aligned} \tag{20}$$

⋆ **Case 4.** In this case we have respectively

$$\begin{aligned} H &= \frac{1}{2}r^4\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r^{-1}\cos\varphi,\\ F_1 &:= rp_\varphi^2\cos\varphi - r^2p_rp_\varphi\sin\varphi + \frac{1}{2}r^{-2}\sin^2\varphi,\\ F_2 &:= r^4\left(p_r^2 - r^{-2}p_\varphi^2\right)\cos\varphi\sin\varphi - r^3p_rp_\varphi\cos(2\varphi) - r^{-1}\sin\varphi. \end{aligned} \tag{21}$$

It appear that also cases 5 and 8 in (15) are integrable.

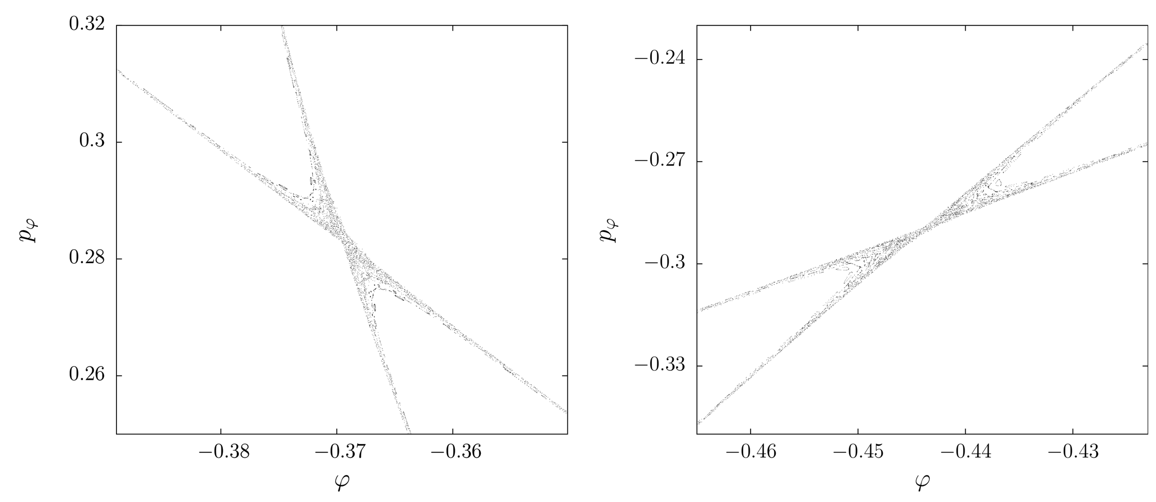
⋆ **Case 5.**

$$\begin{aligned} H &= \frac{1}{2}r\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r^2\cos\varphi,\\ F &:= r^{-1}(p_\varphi^2 - r^2p_r^2)\cos\varphi + r^2(1+\cos^2\varphi) + 2p_rp_\varphi\sin\varphi. \end{aligned} \tag{22}$$

⋆ **Case 8.**

$$\begin{aligned} H &= \frac{1}{2}r^3\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r^{-2}\cos\varphi,\\ F &:= r(p_\varphi^2 - r^2p_r^2)\cos\varphi + r^{-2}(1+\cos^2\varphi) - 2r^2p_rp_\varphi\sin\varphi. \end{aligned} \tag{23}$$

On contrary, it seems that cases 6 and 7 are not integrable, see Fig. 1.



(a) $E = -0.5$ with $m = -2,\; n = -3$ (b) $E = -0.3$ with $m = 2,\; n = 7$

Figure 1: Magnification of region around unstable periodic solution in the Poincaré cross-section plane on the surface $r = 1$

It is important to note that λ also belongs to the first item in the Table 1. However, cases obtained from it are generically non-integrable, see Poincaré section on Fig. 2 showing large chaotic regions.

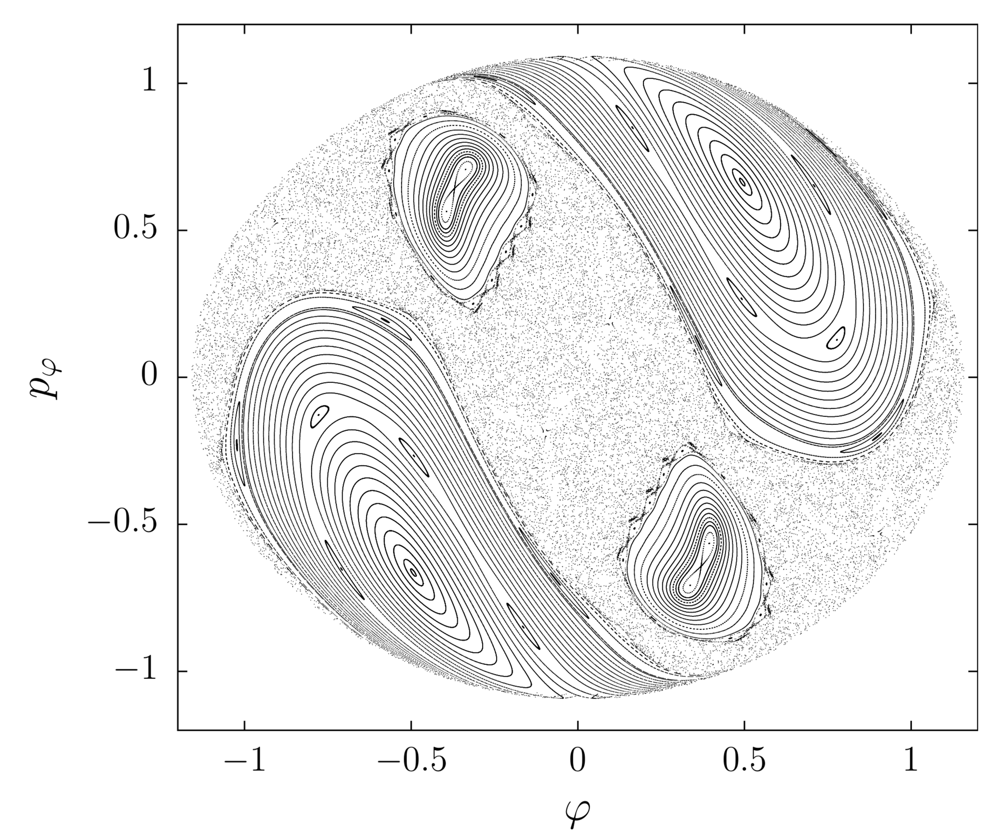


Figure 2: The Poincaré cross sections plane on the surface $r = 1$

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