Killing tensors with non-vanishing Haantjes torsion and integrable systems

A.V. Tsiganov

Saint Petersburg State University

Finite Dimensional Integrable Systems,
Będlewo, July 12-17, 2015

Eigenvalues λ_i of (1,1) tensor K on n-dimensional manifold M with vanishing torsion

$$N(X, Y) = 0$$

are canonical coordinates for K in the domain where they real, distinct, and functionally independent.

In this domain every distribution spanned by the $n-1$ eigenvectors X_i

$$KX_i = \lambda_i X_i\,, \qquad X_i = \frac{\partial}{\partial\lambda_i}\,,$$

is completely integrable:

A. Nijenhuis, *X_{n-1}–forming sets of eigenvectors*, 1951.

How many forming sets of eigenvectors are enough to construct an integrable by Liouville system?

Nijenhuis torsion

For any two commuting (1, 1) tensors A and B Nijenhuis defined a (1, 2) tensor

$$N_{AB}(X, Y) = AB[X, Y] - B[AX, Y] - A[X, BY] + [AX, BY]$$

which is alternating Nijenhuis torsion at $A = B = K$

$$N(X, Y) = K^2[X, Y] - K[KX, Y] - K[X, KY] + [KX, KY]$$

Here X, Y are arbitrary vector fields and $[.,.]$ denotes the commutator of two vector fields.

Other dozen definitions of the Nijenhuis torsion are collected by Boris Kruglikov
http://www.math.uit.no/ansatte/boris/Images/1/12DEF-NJ.pdf

Haantjes torsion

This construction of the integrable distributions and canonical coordinates was slightly modified by Nijenhuis and Haantjes, when they enable to K possess torsion, but in a controlled manner.

In this case condition $N(X,Y)=0$ is replaced by the weaker condition

$$H(X,Y)=0\,,$$

where H is the skew-symmetric $(1,2)$ Haantjes tensor:

$$\begin{aligned} H(X,Y) &= K^2N(X,Y)-KN(KX,Y)-KN(X,KY)\\ &+ N(KX,KY)\,. \end{aligned}$$

Tensor H is a gauge invariant with respect to the gauge transformations of K in the contrast with Nijenhuis tensor N.

The alternating Nijenhuis tensor N and the Haantjes tensor H define deformations of the structures of non-associative and alternating algebras in the tangent bundle TM of M.

As sequence, the Nijenhuis and Haantjes tensors often appear in mathematics and physics, however the overwhelming majority of applications is only related with the simple deformations at

$$N(X,Y)=0 \quad \text{or} \quad H(X,Y)=0\,.$$

- Ferapontov, Marshall, *Differential-geometric approach to the integrability of hydrodynamic chains: The Haantjes tensor*, 2007.
- Bogoyavlenskij, Reynolds, *Criteria for existence of a Hamiltonian structure*, 2010.
- Magri, *Haantjes manifolds*, 2014.
- Tempesta, Tondo, *Haantjes manifolds and integrable systems*, 2014.
- Schöbel, *Nijenhuis integrability for Killing tensors*, 2015.
- Batista, *Integrability Conditions for Killing-Yano Tensors and Conformal Killing-Yano Tensors*, 2015.

Stäckel systems

Let M is a Riemannian manifold with a metric $\mathrm{g}(x)$, which defines the Hamilton function

$$T = \sum_{\alpha,\beta=1}^{n} \mathrm{g}^{\alpha\beta}(x) p_\alpha p_\beta \,.$$

Solutions of the Killing equation

$$\nabla_\alpha K_{\beta\gamma} + \nabla_\beta K_{\gamma\alpha} + \nabla_\gamma K_{\alpha\beta} = 0$$

with simple eigenvalues and vanishing Haantjes torsion

$$H(X, Y) = 0$$

are referred to as the characteristic or integrable Killing tensors.

(real, distinct, functionally independent -> simple eigenvalues)

Stäckel systems

Every integrable Killing tensor K belongs to n-dimensional Killing-Stäckel space of tensors with commuting generators

$$K_1 = \mathrm{g},\;\; K_2 = K\,, \ldots, K_n\,,$$

such that second order polynomials in momenta

$$T_i = \sum_{\alpha,\beta=1}^{n} K_i^{\alpha\beta} p_\alpha p_\beta\,,$$

are in involution

$$\{T_i, T_j\} = 0$$

with respect to the canonical Poisson brackets on T^*M

$$\{p_\alpha, p_\beta\} = \{x_\alpha, x_\beta\} = 0\,, \qquad \{x_\alpha, p_\beta\} = \delta_{\alpha\beta}.$$

Stäckel systems

Then, solving the Bertrand-Darboux type equations we can get separable in a given canonical coordinates potentials $V_i(x)$

$$\{T_i, V_j\} = \{T_j, V_i\}$$

that allow us to construct the necessary number of functionally independent integrals of motion in involution

$$\mathcal{H}_i = T_i + V_i\,, \qquad i = 1, \ldots, n,$$

for the Stäckel systems on T^*M, see Stäckel, Eisenhart, Kalnins, Miller, Benenti, Ibort, Magri etc.

If K is a free-torsion Killing-Yano tensor we can get conserved quantities just enough to enable the explicit integration of the Klein-Gordon and Dirac equations, see Chandrasekhar, Carter, Houri, Kubizňák, Frolov etc.

Our aim

Let M is a Riemannian manifold with the metric $\mathrm{g}(x)$ and geodesic flow is associated with the Hamilton function

$$T = \sum_{\alpha,\beta=1}^{n} \mathrm{g}^{\alpha\beta}(x) p_\alpha p_\beta \,.$$

Let us take solution of the Killing equation

$$\nabla_\alpha K_{\beta\gamma} + \nabla_\beta K_{\gamma\alpha} + \nabla_\gamma K_{\alpha\beta} = 0$$

with non trivial Haantjes torsion

$$H(X, Y) \neq 0$$

and try to add $n-2$ integrals of motion in involution to

$$T_1 = T \quad \text{and} \quad T_2 = \sum_{\alpha,\beta=1}^{n} K^{\alpha\beta}(x) p_\alpha p_\beta$$

Euclidean space

Let $M=\mathbb{R}^3$ is Euclidean space equipped with rectangular Cartesian coordinates $x=(x_1,x_2,x_3)$ and metric $\mathrm{g}=Id$.

Canonical basis of the Killing vectors consists from the translational Killing vectors

$$X_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\qquad X_2=\begin{pmatrix}0\\1\\0\end{pmatrix},\qquad X_3=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

and rotational Killing vectors

$$R_1=\begin{pmatrix}0\\-x_3\\x_2\end{pmatrix},\qquad R_2=\begin{pmatrix}x_3\\0\\-x_1\end{pmatrix},\qquad R_3=\begin{pmatrix}-x_2\\x_1\\0\end{pmatrix}$$

Eisenhart result

Any Killing tensor of valence two is a linear combination of symmetric products of the basic Killing vectors

$$K=\sum_{i\geq j} a^{ij} X_i\odot X_j+\sum_{i,j=1} b^{ij} X_i\odot R_j+\sum_{i\geq j} c^{ij} R_i\odot R_j$$

where 21 coefficients $a^{ij}, b^{ij}, c^{ij}\in\mathbb{R}$ and

$$v_i\odot v_j=\frac{1}{2}\Big(v_i\otimes v_j+v_j\otimes v_i\Big)\,,$$

In 1934 Eisenhart proved that there are only eleven different integrable Killing tensors with vanishing torsion $H=0$.

There is a one-to-one correspondence between integrable Killing tensors, Stäckel systems and orthogonal coordinates in $\mathbb{R}^3$.

What can we say about tensors with $H\neq 0$?

Example

Let us consider the following Killing tensor

$$K = X_1 \odot R_3 + aX_3 \odot R_1 = \begin{pmatrix} -x_2 & \frac{x_1}{2} & 0 \\ \frac{x_1}{2} & 0 & -\frac{ax_3}{2} \\ 0 & -\frac{ax_3}{2} & ax_2 \end{pmatrix} .$$

If $a = -1$, then $H = 0$ and there are two independent eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\left(-x_2 \pm \sqrt{x_1^2 + x_2^2 + x_3^2}\right), \quad \lambda_3 = \lambda_1 + \lambda_2$$

and three second order Stäckel integral of motion

$$T_1 = |p|^2, \quad T_2 = (p \times J)_2, \quad T_3 = \sum_{\alpha,\beta=1}^{n} K_3^{\alpha\beta}(x) p_\alpha p_\beta = J_2^2,$$

where $J = x \times p$ is an angular momentum.

The Hamilton-Jacobi equations are simultaneously separable in parabolic coordinates

$$x_1 = v_1 v_2 \sin\phi, \qquad x_2 = \frac{1}{2}(v_1^2 - v_2^2), \qquad x_3 = v_1 v_2 \cos\phi$$

$$\mathcal{H}_1 = \frac{p_{v_1}^2 + p_{v_2}^2}{v_1^2 + v_2^2} + \frac{p_\phi^2}{v_1^2 v_2^2}, \quad \mathcal{H}_3 = p_\phi^2,$$

$$\mathcal{H}_2 = \frac{v_2^2 p_{v_1}^2 - v_1^2 p_{v_2}^2}{2(v_1^2 + v_2^2)} + \frac{(v_2^2 - v_1^2) p_\phi^2}{2 v_1^2 v_2^2}$$

Angle ϕ is a cyclic coordinate and any linear combination

$$L = \alpha \mathrm{g} + \beta K + \gamma K_3$$

is the Killing tensor with trivial Haantjes torsion $H = 0$.

It is enough to have only one independent forming set of eigenvectors $X_{1,2}$.

Example

Let us consider the same Killing tensor at $a\neq -1$ and $a\neq 0$

$$K=X_1\odot R_3+aX_3\odot R_1=\begin{pmatrix} -x_2 & \dfrac{x_1}{2} & 0\\ \dfrac{x_1}{2} & 0 & -\dfrac{ax_3}{2}\\ 0 & -\dfrac{ax_3}{2} & ax_2 \end{pmatrix}$$

As above we have two second order integrals of motion

$$T_1=p^2\,,\qquad T_2=(p\times J)_2\,,$$

and nothing more in generic case with arbitrary $H\neq 0$.

Main idea: Let two $(n-1)$ eigenvectors $X_{i,j}$ satisfy condition

$$H(X_i,X_j)=0\quad\text{instead of}\quad H\equiv 0$$

Example

At $i \neq j \neq k$ the components of the Haantjes tensor depend on parameter a in the following manner

$$H^1_{23} = -\frac{ax_1x_2x_3}{4}(a+1)(a+2)\,,$$

$$H^2_{13} = \frac{ax_1x_2x_3}{4}(a+1)(a-1)\,,$$

$$H^3_{12} = \frac{ax_1x_2x_3}{4}(a+1)(2a+1)\,,$$

whereas other components only depend on a and $(a+1)$

$$H^1_{13} = -\frac{3ax_1^2x_3}{8}(a+1)\,,\qquad H^2_{21} = \frac{3a^2x_3^2x_1}{8}(a+1)\,.$$

If $a = 1\,, \quad a = -2\,, \quad a = -1/2$ tensor K has three independent complex eigenvalues and eigenvectors

$$\lambda_1(x),\ \lambda_2(x),\ \lambda_3(x) \qquad \text{and} \qquad X_1(x),\ X_2(x),\ X_3(x)$$

which belong to the complexification of the tangent bundle TM. Because

$$H(X_i, X_j) = H_{ij}^k\, X_k\,, \qquad i \neq j \neq k\,,$$

we have three pairs of orthogonal w.r.t. H eigenvectors

$$\begin{array}{lll} 1. & a = \ \ 1, & H(X_1, X_3) = 0\,,\\ 2. & a = -2 & H(X_2, X_3) = 0\,,\\ 3. & a = -\dfrac{1}{2} & H(X_2, X_1) = 0\,. \end{array}$$

Cases $a = -2, -1/2$ are related by permutation $x_1 \leftrightarrow x_3$.

Let us add potentials to the geodesic integrals of motion

$$\mathcal{H}_1 = \sum_{\alpha,\beta=1}^{n} \mathrm{g}^{\alpha\beta}(x) p_\alpha p_\beta + V(x)\,, \quad \mathcal{H}_2 = \sum_{\alpha,\beta=1}^{n} K^{\alpha\beta}(x) p_\alpha p_\beta + U(x)$$

From $\{\mathcal{H}_1, \mathcal{H}_2\} = 0$ follows standard Bertran-Darboux equation

$$d(KdV) = 0\,.$$

At $a = 1$ solution is equal to

$$V = \Big(x_1^4 + 6x_1^2 x_3^2 + x_3^4 + 12x_2^2(x_1^2 + x_3^2) + 16x_2^4\Big)c_1 + \upsilon\,,$$

at $a = -2$ solution is equal to

$$V = \frac{(x_1^2 + 4x_2^2 + 4x_3^2)c_1}{x_1^6} + \upsilon\,,$$

common part of potentials reads as

$$\upsilon = (x_1^2 + 4x_2^2 + x_3^2)c_2 + c_3 x_2 + \frac{c_4}{x_1^2} + \frac{c_5}{x_3^2}\,.$$

Thus, imposing algebraic restrictions on the components of the Haantjes tensor

$$H(X_i, X_j) = 0$$

we found two special Killing tensors with non-zero Haantjes torsion and two nontrivial solutions of the Bertran-Darboux type equation.

To complete the construction of the integrable in the Liouville sense systems the third independent integral of motion in the involution remains to be found.

Fortunately, since first Hamiltonian has been discovered in

B. Dorizzi, B. Grammaticos, J. Hietarinta, A. Ramani, F. Schwarz, *New integrable three-dimensional quartic potentials*, 1986

by using the Yoshida method, **we know how to do it.**

Case $a = 1$

Two quadratic integrals of motion

$$\mathcal{H}_1 = \sum p_i^2 + V(x)\,, \quad \mathcal{H}_2 = p_1 J_3 + p_3 J_1 + U(x)$$

where $J = x \times p$. Potentials

$$V = \left(x_1^4 + 6x_1^2 x_3^2 + x_3^4 + 12x_2^2(x_1^2 + x_3^2) + 16x_2^4\right) c_1 + (x_1^2 + 4x_2^2 + x_3^2)\,c_2 + c_3 x_2 + \frac{c_4}{x_1^2} + \frac{c_5}{x_3^2}$$

$$U = (x_1^2 - x_3^2)\left(2x_2(x_1^2 + 2x_2^2 + x_3^2)\,c_1 + x_2 c_2 + \frac{c_3}{4}\right) - \frac{c_4 x_2}{x_1^2} + \frac{c_5 x_2}{x_3^2}$$

Fourth order integral of motion

$$\mathcal{H}_3 = T_3 T_4 + \sum_{\alpha,\beta=1}^{n} S^{\alpha\beta}(x) p_\alpha p_\beta + W(x)$$

where

$$T_3 = p_1^2\,, \qquad T_4 = p_3^2$$

Case $a = -2$

Two quadratic integrals of motion

$$\mathcal{H}_1 = \sum p_i^2 + V(x), \quad \mathcal{H}_2 = p_1 J_3 - 2p_3 J_3 + U(x)$$

where $J = x \times p$. Potentials

$$V = \frac{(x_1^2+4x_2^2+4x_3^2)c_1}{x_1^6} + (x_1^2+4x_2^2+x_3^2)c_2 + c_3 x_2 + \frac{c_4}{x_1^2} + \frac{c_5}{x_3^2}$$

$$U = -\frac{2c_1 x_2(x_1^2+2x_2^2+2x_3^2)}{x_1^6} + c_2 x_2(x_1^2+2x_3^2) + \frac{c_3(2x_3^2+x_1^2)}{4} - \frac{2c_5 x_2}{x_3^2} - \frac{c_4 x_2}{x_1^2}$$

Fourth order integral of motion

$$\mathcal{H}_3 = T_3 T_4 + \sum_{\alpha,\beta=1}^{n} S^{\alpha\beta}(x) p_\alpha p_\beta + W(x)$$

where

$$T_3 = p_1^2, \qquad T_4 = J_2^2 + J_3^2 .$$

For the Stäckel systems we have three Killing tensors K_1, K_2, K_3 with vanishing Haantjes torsion $H = 0$ so that

$$\{T_i, T_j\} = 0, \qquad T_i = \sum_{\alpha,\beta=1}^{3} K_i^{\alpha\beta} p_\alpha p_\beta .$$

For the new systems we have four Killing tensors K_1, K_2, K_3 and K_4 so that

$$\{T_1, T_k\} = 0, \qquad \{T_2, T_3\}T_4 + \{T_2, T_4\}T_3 = 0,$$

among them three tensors $K_1 = \mathrm{g}$, K_3, K_4 with $H = 0$ and one Killing tensor K_2 with $H \neq 0$.

Then we have to add potentials using Bertrand-Darboux type equations.

Every embedded smooth submanifold inherits a metric from being embedded in the Riemannian manifold, and every covering space inherits a metric from covering the Riemannian manifold.

Pushforward and pullback of other tensors are well defined too.

The pullback of the non integrable Killing tensor K to plane $\mathbb{R}^2$ either reduces it to integrable Killing tensor, or annihilates it.

At $a=1$ and $x_3=p_3=0$

$$\mathcal{H}_1^{(r)}=p_1^2+p_2^2+2c_1(x_1^4+6x_1^2x_2^2+8x_2^4)+c_2(x_1^2+4x_2^2)+c_3x_2+\frac{4c_4}{x_1^2}$$

at $a=-2$ and $x_2=p_2=0$

$$\mathcal{H}_1^{(r)}=p_1^2+p_3^2+\frac{c_1(x_1^2+4x_3^2)}{x_1^6}+c_2(x_1^2+x_3^2)+\frac{c_4}{x_1^2}+\frac{c_5}{x_3^2}$$

Second integrals of motion are fourth order polynomials.

Conclusion:

The modern geometric theory of Killing and Killing-Yano tensors without torsion provides a wide variety of tools that allow us:

- to construct symmetries and integrals of motion;
- to get variables of separation and bi-Hamiltonian structures;
- to establish relations between various integrable systems;
- to obtain Lax matrices and classical r-matrices;
- to integrate equations of motion etc.

However, the theory is not far enough advanced to offer similar tools for the Killing and Killing-Yano tensors with nontrivial Haantjes torsion.