Two dimensional superintegrable systems all of whose geodesics are closed

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Bedlewo July 13-17, 2015

Outline

Integrability versus superintegrability

In a configuration space of dimension n, the set of classical observables

$$\widetilde{\mathcal{O}} = \mathcal{O} \cup \{S_1, \ldots, S_\nu\} \qquad \mathcal{O} = \{H, Q_1, \ldots, Q_{n-1}\}$$

is said to be superintegrable (SI) if:

- The subset $\mathcal{O}$ is integrable.
- One has $\{H, S_i\} = 0 \quad i = 1, \ldots \nu \leq n-1$
  (maximally SI if $\nu = n-1$)
- All the elements of $\widetilde{\mathcal{O}}$ are functionally independent.

A recent review may be found in

W. Miller Jr, S. Post and P. Winternitz
Classical and quantum superintegrability with applications
J. Phys. A, Math. Theor. **46** 423001-98 (2013)

Preliminaries (1)

In the research of SI systems a lot of results were obtained for metrics g of constant curvature mainly on surfaces. The first effort to free oneself from this restriction is due to Koenigs in

note in G. Darboux, "Leçons sur la Théorie des Surfaces", vol. 4, Chelsea Publishing (1972) 368-404.

He started from the metric

$$g = \frac{dx^2 + dy^2}{h_x^2} \quad \Longleftrightarrow \quad H = h_x^2(P_x^2 + P_y^2)$$

where $h = h(x)$ and $h_x = D_x\, h(x)$.
Since $\{H, P_y\} = 0$ this system is integrable.
He went on to get a (maximal, since we are in dimension two) SI system by looking for an extra integral Q quadratic in the momenta.

Preliminaries (2)

He was able to solve this problem, to give a finite list of metrics of non constant curvature and even exhibit, for each metric, two quadratic integrals Q_1 and Q_2 so that the span of quadratic observables $\mathcal{I}^2(g)$ is four dimensional with basis

$$H \qquad P_y^2 \qquad Q_1 \qquad Q_2.$$

However, as shown in

E. G. Kalnins, J. M. Kress, W. Jr. Miller, P. Winternitz
Superintegrable systems in Darboux spaces
J. Math. Phys **44** (2003) 5811.

in the quadratic algebra

$$\{Q_1, Q_2\} = P_y(a_1\, H + a_2\, P_y^2) \qquad Q_1^2 + Q_2^2 = c_1\, H^2 + c_2\, H\, P_y^2 + c_3\, P_y^4$$

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the last relation shows that Q_1 and Q_2 are not functionally independent. An important shortcoming was that none of the metrics discovered by Koenigs could be defined on a closed manifold.

Matveev and Shevchishin construction (1)

To cure this drawback Matveev and Shevchishin proposed a generalization of Koenigs construction in

"2-dim SI metrics with one linear and one cubic integral"

J. Geom. and Phys. **61** (2012) 1353-1377

They considered a SI system with an extra *cubic* integral

$$S = a_1(x,y)\,P_x^3 + a_2(x,y)\,Px^2\,P_y + a_3(x,y)\,P_x\,P_y^2 + a_4(x,y)\,P_y^3$$

with linear span $\mathcal{I}^3(g)$. Imposing $\{H,S\} = 0$ gives a system of coupled PDE quite difficult to solve.

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Key idea: consider, in $\mathcal{I}^3(g)$, the linear endomorphism L defined as

$$L : \; S \to \; \{P_y, S\}$$

and classify the cubic integrals according to its eigenvalues λ. We have:

$$\{P_y, S\} = \lambda\, S \quad \Longrightarrow \quad a_k(x,y) = a_k(x)\; e^{\lambda\, y} \qquad k = 1, \ldots, 4$$

Matveev and Shevchishin construction (2)

The requirement $\{H, S\} = 0$ gives a set of coupled non linear ODE for the unknown function h and a detailed analysis of these equations led Matveev and Shevchishin to split their analysis into 3 cases:

$$(i): \quad \lambda = \mu \qquad (ii): \quad \lambda = i\,\mu \qquad (iii): \quad \lambda = 0 \qquad\qquad (\mu \in \mathbb{R})$$

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$$(i): \quad \lambda = \mu \qquad (ii): \quad \lambda = i\,\mu \qquad (iii): \quad \lambda = 0 \qquad\qquad (\mu \in \mathbb{R})$$

and, accordingly, h must be a solution of:

$$(i): \; h_x(A_0\, h_x^2 + \mu^2 A_0\, h^2 - A_1\, h + A_2) = A_3 \frac{\sinh(\mu x)}{\mu} + A_4\, \cosh(\mu x)$$

$$(ii): \; h_x(A_0\, h_x^2 - \mu^2 A_0\, h^2 - A_1\, h + A_2) = A_3 \frac{\sin(\mu x)}{\mu} + A_4\, \cos(\mu x)$$

$$(iii): \; h_x(A_0\, h_x^2 - A_1\, h + A_2) = A_3\, x + A_4 \qquad \text{(limit as } \mu \to 0)$$

A beautiful result, reminiscent of the spaces with constant curvature.

Matveev and Shevchishin construction (3)

Once h is known, considering for instance the case (i), two cubic integrals were obtained

$$\mathcal{S} \equiv S_1 + iS_2 = e^{-iy}(\mathcal{A} + i\,\mathcal{B}) \qquad\qquad H = h_x^2(P_x^2 + P_y^2)$$

with

$$\mathcal{A} = P_x\Big(a_0(x)\,P_x^2 + a_1(x)\,P_y^2\Big) \qquad\qquad \mathcal{B} = P_y\Big(a_2(x)\,P_x^2 + a_3(x)\,P_y^2\Big)$$

and Matveev and Shevchishin were able to express the $a_i(x)$ in terms of h and its derivatives.

It follows that $\mathcal{I}^3(g)$, as $\mathcal{I}^2(g)$ in Koenigs case, is 4 dimensional with basis

$$P_y^3 \qquad\qquad H\,P_y \qquad\qquad S_1 \qquad\qquad S_2.$$

Matveev and Shevchishin construction (4)

Remarks:

- Shevchishin had also obtained the following relations

$$\{S_1, S_2\} = P_y(a_1\, H^2 + a_2\, H\, P_y^2 + a_3\, P_y^4)$$

$$S_1^2 + S_2^2 = c_1\, H^3 + c_2\, H^2\, P_y^2 + c_3\, H\, P_y^4 + c_4\, P_y^6$$

where all the a_i and c_i are constants. The second relation shows, as in Koenigs case, the functional dependence of S_1 and S_2 which allows to consider two different SI models:

$$\mathcal{I}_1 = (H,\ P_y,\ S_1) \qquad \text{or} \qquad \mathcal{I}_2 = (H,\ P_y,\ S_2).$$

- Since we have $a_0 = A_0\, h_x^3$, in the limit $A_0 \ \rightarrow\ 0$ the cubic integrals become reducible $S_{1,2} = P_y\, Q_{1,2}$, with $\{H, Q_{1,2}\} = 0$ and we recover the quadratically SI models of Koenigs.

Integration modulo diffeomorphisms (1)

The integration of the three ODE was effected in

G. V., C. Duval and V. Shevchishin

"Explicit metrics for a class of two-dimensional SI systems"

J. Geom. Phys. **87** (2015) 461-481

I will focus on the most interesting case (i):

$$h_x(A_0\,h_x^2+\mu^2 A_0\,h^2-A_1\,h+A_2)=A_3\,\frac{\sinh(\mu x)}{\mu}+A_4\,\cosh(\mu x)$$

simplified by the following choices:

$$A_0=1\ (\neq\ \text{Koenigs})\qquad \mu=1\ (\text{scaling } x)\qquad A_1=0\ (\text{translating } h)$$

We are left with

$$h_x(h_x^2-h^2-a)=\frac{\lambda}{2}(e^x+\epsilon\,e^{-x})\qquad \epsilon=0,\pm1\qquad (a,\lambda)\in\mathbb{R}^2.$$

Integration modulo diffeomorphisms (2)

and we have to solve in fact two problems:

1. The local problem of finding coordinates for which the metric will be explicit. The key is not to take seriously the initial coordinate x! We have to use local diffeomorphisms if we want to produce an explicit metric.

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2. The global problem: the local analysis, leading to the solution $h(x)$, does involve a finite number of real parameters. We have to determine for which values of these parameters it is possible for the metric

$$g = \frac{dx^2 + dy^2}{h_x^2}$$

   and its integrals $S_{1,2}$ to be defined on a manifold.

Solving the local problem.

Let us define:

$$u = h_x \qquad v = u^2 - h^2 \qquad U = u(u^2 - h^2 - a) = \frac{\lambda}{2}(e^x + \epsilon\, e^{-x}).$$

We have a first ODE

$$U_{xx} - U = 0 \qquad \Longrightarrow \qquad D_h\Big(u\, D_h\, U\Big) = v - a$$

which can be integrated and gives

$$4hu\, D_h\, U = c + (v - a)(3v^2 + 4h^2 + a) \qquad\qquad c \in \mathbb{R}.$$

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And we have also

$$U_x^2 = U^2 - \epsilon\lambda^2 \quad \Longrightarrow \quad \Big(4hu\, D_h\, U\Big)^2 = 16h^2[v(v-a)^2 - \epsilon\lambda^2 + (v-a)^2 h^2]$$

Combining we get

$$[c + (v-a)(3v^2 + a) + 4(v-a)h^2]^2 = 16h^2[v(v-a)^2 - \epsilon\lambda^2 + (v-a)^2 h^2]$$

which is linear in h^2! We will solve for $h^2(v)$ and v is the appropriate coordinate we were looking for.

Final form of the metric

One obtains:

$$h^2 = \frac{D'^2}{8D} \qquad\qquad D = -(v-a)(v^2 - a^2 + c) - 2\epsilon\lambda^2.$$

Transforming the metric to the new coordinates (v, y) we get

The metric of the SI system

The local form of the metric is

$$\frac{g}{2} = \left(\frac{Q}{P}\right)^2 \frac{dv^2}{D} + \frac{4D}{P} dy^2$$

with the quartic polynomials

$$P = 8vD + (D')^2 \qquad\qquad Q = 2DD'' - (D')^2.$$

The cubic polynomial $D(v)$ is driving the metric with highest order term $-v^3$ (and v^3 in the case (ii)).

Remarks:

- The local structure of the metric is under control of the cubic polynomial D and for the metric to be riemannian we must impose

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$$\frac{dx}{dv} = \frac{Q}{2D\sqrt{P}}.$$

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- Shevchishin algebraic relations are explicitly:

$$\{S_1, S_2\} = P_y(-8a\,H^2 - 4cHP_y^2 + 6\epsilon\lambda^2\,P_y^4)$$

$$S_1^2 + S_2^2 = 8\,H^3 + 8a\,H^2 P_y^2 + 2c\,HP_y^4 - 2\epsilon\lambda^2\,P_y^6$$

Looking for a manifold (1)

Since we look for simply connected riemannian manifolds, by the uniformization theorem they must be conformally related to the spaces of constant curvature.
The generic situation is the following: the constraints $D > 0$ and $P > 0$ allow for the coordinate v some open interval I = (a, b). A priori the end-points are singularities of the metric and we have to determine whether they are true singularities (scalar curvature singularities, conical singularities...) or just apparent singularities (coordinates singularities).

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Let us consider a first example:

$$g = d\theta^2 + \sin^2\theta\, d\phi^2 \qquad \theta \in\ (0, \pi) \quad \phi \in \mathbb{S}^1.$$

Obviously $\theta = 0$ is a singularity of this metric. Let us show that it is an apparent singularity.

Looking for a manifold (2)

In a neighbourhood of $\theta = 0$, using the new coordinates (x, y) defined by

$$x = 2\,\frac{\sin\theta}{1+\cos\theta}\,\sin\phi \qquad y = 2\,\frac{\sin\theta}{1+\cos\theta}\,\cos\phi$$

the metric becomes

$$g = \frac{dx^2 + dy^2}{\left(1 + \frac{x^2+y^2}{4}\right)^2}$$

which is indeed C^∞ in a neighbourhood of the origin.
Geometrically speaking we have discovered the north pole of a sphere! A similar argument will work for $\theta = \pi$, the south pole.

Each pole, by the index theorem, increases by 1 the Euler characteristic hence we have $\chi = 2$ and this metric is indeed defined on $\mathbb{S}^2$ (since, as is well known, the initial g is the canonical embedding of the sphere in $\mathbb{R}^3$!).

Tannery orbifold

Let us consider now the metric:

$$g = (2 + \cos\theta)^2\, d\theta^2 + \sin^2\theta\, d\phi^2 \qquad \theta \in\ (0, \pi) \quad \phi \in \mathbb{S}^1.$$

for which $\theta = \pi$ is a smooth south pole. However for $\theta = 0$ we have

$$\frac{g}{9} \approx d\theta^2 + \sin^2\theta\, \left(\frac{d\phi}{3}\right)^3$$

and now this point is a true singularity since $\phi/3 \notin \mathbb{S}^1$. It is called a conical singularity. Hence this metric is not defined on a manifold but rather on an orbifold (Thurston weakening of the concept of manifold).

For reasons to be clear later on we will speak of Tannery orbifold $O = \mathcal{T}^2$.

A globally defined metric on the two sphere (1)

By a case by case study we have proved:

Theorem 2 (in the quoted article)

For $l \in [-1, +\infty)$ and $m > 1$ the metric

$$g = \left(\frac{Q}{P}\right)^2 \frac{dx^2}{D} + \frac{4D}{P}\, d\phi^2 \qquad x \in (-1, +1) \qquad \phi \in \mathbb{S}^1$$

with

$$\left\{\begin{array}{l} D = (x+m)(1-x^2) \qquad\qquad L_\pm = l \pm \sqrt{l^2 - 1} \\[2ex] P = \Big(L_+(1-x^2) + 2(m+x)\Big)\Big(L_-(1-x^2) + 2(m+x)\Big) \\[2ex] Q = 3x^4 + 4mx^3 - 6x^2 - 12mx - 4m^2 - 1 \end{array}\right.$$

is globally defined on $M \cong \mathbb{S}^2$.

A globally defined metric on the two sphere (2)

Let us check the end points $x = \pm 1$ by setting $x = \cos\theta$ with $\theta \in\ (0,\,\pi)$. We have

$$g(\theta \to 0+) \approx \frac{1}{m+1}(d\theta^2 + \sin^2\theta\, d\phi^2)$$

and

$$g(\theta \to \pi-) \approx \frac{1}{m-1}(d\theta^2 + \sin^2\theta\, d\phi^2).$$

The Euler characteristic can be computed and gives $\chi = 2$ proving that $M \cong \mathbb{S}^2$.

Some further work is needed to check that the two integrals S1 and S2 are also globally defined.

Zoll and Tannery metrics (1)

Matveev and Shevchishin were interested in getting SI systems on either $\mathbb{S}^2$ or $P^2(\mathbb{R})$, because they expected to obtain Zoll metrics, for which all of the geodesics are closed and of the same length 2π.

K. Kiyohara provided us with a detailed but unpublished proof valid for Matveev and Shevchishin model.

We found it interesting to give an explicit proof, relying on the following result, given in:

A. Besse

Manifolds all of whose geodesics are closed

Springer-Verlag, Berlin (1978)

Zoll and Tannery metrics (2)

Besse canonical form

Any metric of the form

$$g = A^2(\theta)\, d\theta^2 + \sin^2\theta\, d\phi^2 \qquad \theta \in\ (0, \pi) \qquad \phi \in \mathbb{S}^1$$

for which A is given by

$$A(x) = \frac{p}{q} + a(x) \qquad\qquad x = \cos\theta$$

where p and q are integers and $a(x)$ is an odd function of x is a (p, q) Tannery metric and all of its geodesics are closed.

Zoll and Tannery metrics (3)

For Tannery metrics:

- The equator has for length 2π.
- The meridians have for length $2\,\frac{p}{q}\,\pi$.
- All of the remaining geodesics have for length $2p\pi$.

For Zoll metrics:

- We have $p = q = 1$ hence all the geodesics have one and the same length length 2π.
- They are also called $C_{2\pi}$-metrics.
- The simplest one being the round metric on $\mathbb{S}^2$.

Zoll metrics

Let us examine the results obtained in

G. V.
Zoll and Tannery metrics from a SI geodesic flow
Lett. Math. Phys. 104 (2014) 1121-1135

We have first

Theorem

The SI metric defined on $M = \mathbb{S}^2$ is a Zoll metric.

Indeed the change of coordinate

$$x = 1 + 2\frac{\cos\theta - H_-}{H_+ + H_-} \qquad H_\pm(\theta) = \sqrt{1 - \frac{(l \pm 1)}{(l+m)}\sin^2\theta} \quad l + m > 0$$

gives Besse canonical form with

$$A(\theta) = 1 + \cos\theta\left(\frac{1}{H_+} - \frac{1}{H_-}\right).$$

Tannery pear (1)

This is not the whole story. If we take

$$D(v) = v^2(1-v) \qquad\qquad v \in \ (-\infty, 1)$$

the first change of coordinate $w = 1 - v$ gives

$$g = \frac{(1+3w)^2}{(1+w)^4}\, dw^2 + \frac{4w}{(1+w)^2}\, d\phi^2 \qquad w \in \ (0, +\infty) \qquad \phi \in \mathbb{S}^1$$

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and the second change of coordinate

$$w = \frac{1+\cos\theta}{1-\cos\theta} \qquad\qquad \theta \in (0,\, \pi)$$

leads us to Besse canonical form:

$$g = (2+\cos\theta)^2\, d\theta^2 + \sin^2\theta\, d\phi^2$$

which we gave as an example of a metric defined on an orbifold.

Tannery pear (2)

Remarks:

- We have $p = 2$ and $q = 1$ Tannery metric: all the geodesics have the same length 4π but the equator which is of length 2π.
- This metric describes a surface which is called Tannery pear. It was discovered in 1892 by Jules Tannery who was able to integrate the geodesic equations. It is why we called the corresponding orbifold $O = \mathcal{T}^2$ Tannery orbifold.
- He was an expert in elliptic function theory (a famous book with Jules Molk) and in the philosophy of sciences.
- Our analysis suggests that his ability to integrate the geodesic equations has its root in the fact that the geodesic flow is SI.

Jules Tannery (1848-1910)

Geometry of Tannery pear

Some properties of this orbifold:

- Its measure and its sectional curvature are

$$\mu(\mathcal{T}^2, g) = 8\pi \qquad\qquad \sigma(\mathcal{T}^2, g) = \frac{2}{(2+\cos\theta)^3}.$$

- It can be (globally) embedded in $\mathbb{R}^3$ according to

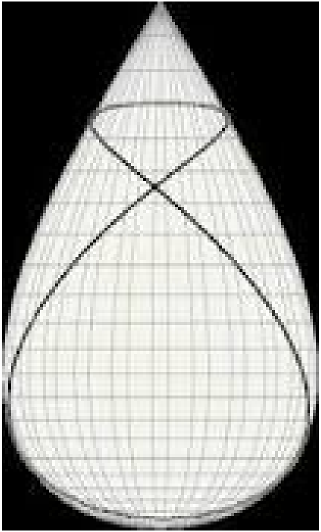
$$X = \sin\theta\,\cos\phi \qquad Y = \sin\theta\,\cos\phi \qquad Z = 4\sqrt{2}\sin\frac{\theta}{2}$$

giving a quartic for its cartesian equation

$$X^2 + Y^2 = \frac{Z^2}{8}\left(1 - \frac{Z^2}{32}\right) \qquad\qquad Z \in [0, 4\sqrt{2}].$$

The point $(X, Y, Z) = (0, 0, 4\sqrt{2})$ is a smooth pole while the point $(0, 0, 0)$ is the vertex of a cone with an aperture of $2\arctan\left(\frac{1}{2\sqrt{2}}\right)$ close to 39°.

Tannery pear and a geodesic



Further metrics on Tannery orbifold

We have obtained some generalizations, still living on the same orbifold $O = \mathcal{T}^2$:

- A simple one parametric extension with

$$A(\theta) = 2 + \frac{\cos\theta}{R(\theta)} \qquad R(\theta) = \sqrt{1 + \rho\sin^2\theta} \qquad \rho > 0$$

  which gives Tannery pear for $\rho \to 0$.
- A more complicated two parametric extension which is Zoll and no longer Tannery as the previous one.

Geodesic equations (1)

Starting from Besse canonical form of the metric

$$g = A^2(\theta)\, d\theta^2 + \sin^2\theta\, d\phi^2 \quad \Rightarrow \quad H = \frac{1}{2}\left(\Pi^2 + \frac{P_\phi^2}{\sin^2\theta}\right) \quad \Pi = \frac{P_\theta}{A(\theta)}.$$

Let us consider the invariant torus

$$H = E \in \mathbb{R} \qquad\qquad P_\phi = L > 0$$

for which the Hamilton equations give

$$\sin^2\theta\, \frac{d\phi}{dt} = L > 0 \qquad \Pi = \epsilon\sqrt{2E - \frac{L^2}{\sin^2\theta}} \qquad \epsilon = \pm 1.$$

Geodesic equations (2)

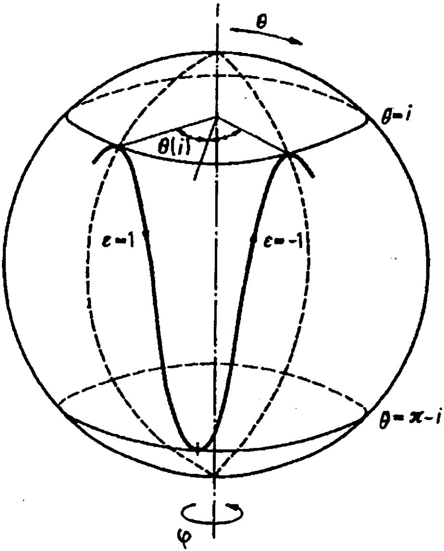
Let us add some remarks:

- If the sign is $\epsilon = -1$ (resp. $\epsilon = +1$) the angle θ is decreasing (resp. increasing).
- The choice $2E = 1$ identifies the time coordinate t with the arc length s.
- We will take for initial conditions

$$\theta = i \in \left(0, \frac{\pi}{2}\right) \qquad \phi = 0 \qquad L = \sin i \qquad \Pi = 0.$$

The following figure will help:

Besse geometric setting for the geodesics



The cubic integrals give the geodesic equations (1)

For Tannery pear the cubic integrals are

$$S_1 = +\cos\phi\,\Pi\,(1 + L^2\,\alpha) + \sin\phi\,L\,(\beta + L^2\,\gamma)$$

$$S_2 = -\sin\phi\,\Pi\,(1 + L^2\,\alpha) + \cos\phi\,L\,(\beta + L^2\,\gamma)$$

with

$$\alpha(\theta) = -\frac{(1+c)^2}{s^2} \qquad \beta(\theta) = -\frac{(1+2c)}{s} \qquad \gamma(\theta) = \frac{(1+c)^2}{s^3}$$

where $s = \sin\theta$ and $c = \cos\theta$.

The initial values of the integrals are:

$$S_1 = 0 \qquad\qquad S_2 = \cos^2 i$$

The cubic integrals give the geodesic equations (2)

From these two conservation laws we extract the geodesic equations

$$\sin\phi = -\Pi(\theta)\Big(1+\tan^2 i\,(1+\alpha(\theta))\Big)$$

$$\cos\phi = \sin i\Big(\beta(\theta)+\tan^2 i\,(\beta(\theta)+\gamma(\theta)\Big)$$

$$\Pi(\theta) = \epsilon\sqrt{1-\frac{\sin^2 i}{\sin^2\theta}}$$

The cubic integrals give the geodesic equations (2)

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$$\sin\phi = -\Pi(\theta)\Big(1 + \tan^2 i\,(1 + \alpha(\theta))\Big)$$

$$\cos\phi = \sin i\Big(\beta(\theta) + \tan^2 i\,(\beta(\theta) + \gamma(\theta)\Big)$$

$$\Pi(\theta) = \epsilon\sqrt{1 - \frac{\sin^2 i}{\sin^2\theta}}$$

Moving along a geodesic we have:

1. Initially we have $\theta = i$ and $\phi = 0$.
2. As θ decreases to $\frac{\pi}{2}$ the angle ϕ increases to $\pi + i$.
3. For $\theta = \pi - i$ we have $\phi = 2\pi$.
4. In the way back from $\theta = \pi - i$ to $\theta = i$ the geodesic will close, in two turns. Its length will be 4π since the two halves of the geodesic are symmetric.

Generalization to the quartic case (1)

The case of an extra integral quartic in the momenta was undertaken for his Thesis by P. Novichkov with V. Shevchishin as supervisor. Their results cover the three cases (i), (ii) and (iii). Focusing on the case (i), as previously:

- The quartic integrals are now

$$\mathcal{S} \equiv S_1 + iS_2 = e^{-iy}(\mathcal{A} + i\,\mathcal{B}) \qquad\qquad H = h_x^2(P_x^2 + P_y^2)$$

  with

$$\mathcal{A} = a_0(x)\,P_x^4 + a_2(x)\,P_x^2\,P_y^2 + a_4(x)\,P_y^4$$

  and

$$\mathcal{B} = P_x\,P_y\left(a_1(x)\,P_x^2 + a_3(x)\,P_y^2\right)$$

  and the detailed expressions of the $a_i(x)$ in terms of h and its derivatives have been obtained, as in the cubic case.
- In the limit $a_0 \to 0$ one recovers the cubic case.

Generalization to the quartic case (2)

- Setting, as in the cubic case:

$$\mu = 1, \quad A_0 = 1, \quad A_1 = 0 \qquad \phi(x) = A_3 \cosh x + A_4 \sinh x$$

the master function h is now an *algebraic* solution of a set of two ODE:

$$2\phi\, h\, h_x^3 - [(h^2 - 2A_2)\phi' - A_5]h_x^2 - \phi^2 = 0$$

and

$$4h^2\, h_x^3 - [h^2(h^2 - 4A_2) - 8\phi' + A_6]h_x - 8h\,\phi = 0.$$

- The algebraic relations

$$\{S_1, S_2\} = P_y \sum_{l=0}^{3} a_l\, H^l\, P_y^{6-2l} \qquad S_1^2 + S_2^2 = \sum_{l=0}^{4} c_l\, H^l\, P_y^{8-2l}$$

still hold.

- Last but not least, the existence of globally defined metrics has been proved.

Outlook and open problems

Concluding remarks:

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- What can be done for higher degrees? We are still lacking globally defined examples with integrals of degree strictly higher than 4 in the momenta.
- In the cubic case, we observed that for a SI geodesic flow on a closed manifold the metric is Zoll. Does this survive for the quartic and possibly higher degrees?

Terras Incognitas:

- Still in the cubic case it seems to be the first time that an orbifold creeps into the field of superintegrability. As we have seen it may stand either Zoll or Tannery metrics. Is this specific to the cubic case or could it be generalized to the quartic and possibly higher degrees?

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- A recurrent question is also: could one find some quantization scheme which would preserve the superintegrability at the quantum level, at least in the case of cubic integrals. Carter minimal quantization, successful for a large class of quadratically integrable models, does not work any longer!