Determining the Number of Polynomial Integrals

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The problem and how we approach it

Question: How many smooth integrals polynomial in momenta does an (explicitly) given metric admit?

Plan of the talk:

- There is an algorithmic way to do it (description to follow later)
- This way of approach can be realised on a computer, and has been applied for the following examples:
  - Zipoy-Voorhees metric (with parameter $\delta = 2$)
    (result: nonexistence of an additional integral up to degree 11)
  - Weyl's class metric (static and axisymmetric Ricci-flat metrics)
    (result: nonexistence of an additional integral of degree 3)
  - Sub-Riemannian structures on Carnot groups (with B. Kruglikov, G. Lukes-Gerakopoulos)
    (result: nonexistence of an additional integral up to degree 5 resp. 6 for 3 examples in dimensions 6, resp. 7 and 8)

- The condition for existence of an integral polynomial in momenta can be formulated as a system of PDE on the components of the integral. Let $I = K^{i_1,\dots,i_d} p_{i_1} \cdots p_{i_d}$. The requirement for I to be an integral is

$$\nabla_{(j} K_{i_1,\dots,i_d)} = 0$$

(this is equivalent to the equations obtained from $\{H, I\} = 0$ with $H = g^{ij} p_i p_j$).
- In the simplest situation the metric is explicitly given
- The PDE system on components of Killing tensor is overdetermined and of finite type (to be explained).
- There exists a classical, though computationally hard method to deal with such systems (i.e. prolongation-projection method). This method can be implemented on a computer (and we do this). It determines the space of integrals (in involution with known ones) polynomial in momenta of fixed degree d.

  Two integrals I_1, I_2 are in involution if $\{I_1, I_2\} = 0$.

Idea of the Projection-prolongation algorithm:

- Consider a linear overdetermined system of PDE

$$\begin{pmatrix} \textit{coefficients} \end{pmatrix} \begin{pmatrix} \textit{un-} \\ \textit{knowns} \\ \vdots \end{pmatrix} = 0$$

  The coefficients are determined be the metric and its derivatives. The unknowns are the unknown functions ($K^{i_1,\ldots,i_d}$ in the integral $I = K^{i_1,\ldots,i_d} p_{i_1} \cdots p_{i_d}$) and their derivatives
- Derivatives are taken of this system of equations w.r.t. all coordinates, and the newly obtained equations are added to the system of equations. Derivatives of the unknown functions are treated as independent unknowns.
- If the PDE system is overdetermined and of finite type, the number of unknowns will be less than the number of equations after a finite number of derivation steps.

$$\begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} \begin{pmatrix} \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

- The matrix rank can be calculated using computer algebra (if we choose a generic point). If the rank is big enough, then the initial PDE system has no non-trivial solutions. Computationally, there is a difficulty in calculating the rank, because the matrix dimensions are huge:

Example	Equations	Unknowns
Zipoy-Voorhees degree 11	10780	10192
Sub-Riemannian (2,3,5,8) degree 5	48048	28512

- The method works for arbitrary degree but additional tricks may be needed for application to higher degrees.
  - Decomposition of the PDE system into separate subsystems
  - Partial solution of the matrix system using simple equations (e.g. monomial ones)

First and Second Example: Stationary and Axially Symmetric Space-Times

Weyl's class

Most simple example: Zipoy-Voorhees Metrics

Pseudo-Riemannian Metric:

$$g = \left(\frac{x+1}{x-1}\right)^{\delta} \left(\left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2-1} \left(dx^2 + \frac{x^2-1}{1-y^2}\, dy^2\right) \right.$$
$$\left. +(x^2-1)(1-y^2)d\phi^2 \right) - \left(\frac{x-1}{x+1}\right)^{\delta} \; dt^2$$

Parameter value $\delta = 0$ in flat case, $\delta = 1$ for the Schwarzschild metric.

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Theorem

There is no additional irreducible and involutive integral polynomial in momenta in degree up to 11 for $\delta = 2$.

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- Nonexistence up to degree 6 for $\delta = 2$ (Kruglikov & Matveev, 2011)
- Poincaré section analysis found evidence of non-integrability for several values of δ (Lukes-Gerakopoulos, 2012)
- Nonexistence of meromorphic integrals for several values of δ (Maciejewski & Przybylska & Stachowiak, 2013)

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Degree	...	5	6	7	8	...	11
Equations	...	630	1120	1980	3150	...	10780
Unknowns	...	560	1080	1800	3025	...	10192
Comp. time	...	16sec	1.3min	8min	35min	...	12days

Second Example: Nonexistence of Degree-3 Integrals for Weyl Metrics

Definition:

A 4-dimensional metric is Weyl if it is Ricci-flat, static and axisymmetric. Such metrics can be written locally in the following form:

$$g = e^{2U(x,y)} \left(e^{-2\gamma(x,y)} \left(dx^2 + dy^2 \right) + x^2 \, d\phi^2 \right) - e^{-2U(x,y)} dt^2$$

- Ricci-flatness implies the so-called Ernst Equations for U and γ. They express derivatives of γ by derivatives of U.
- Weyl metrics always admit the Killing vectors $\frac{\partial}{\partial \phi}$ and $\frac{\partial}{\partial t}$. We look for additional irreducible integrals in involution with these and the Hamiltonian.

Does a Weyl metric admit an additional irreducible and involutive integral of third degree in momenta?

Theorem:

There is no additional, irreducible and involutive, third-degree integral polynomial in momenta for metrics in Weyl's class.

Idea of the Proof:

The metric is determined by one function, $U(x, y)$. At first, let us consider this function given.

Apply prolongation-projection and obtain a PDE system for U (possible by hand).

Then apply prolongation-projection again for this system, including the Ernst equations, and show that there are no solutions for this system (computer algebra needed).

(arXiv:1506.06926)

Third Example: Sub-Riemannian Structures on Carnot Groups

joint with Boris Kruglikov (Tromsø) and
Georgios Lukes-Gerakopoulos (Prague)

(arXiv:1507.03082)

The Problem

The problem has already been described by Boris Kruglikov in his talk. Here we only present the application of the algorithm.

With the algorithm, we showed non-existence of an additional (final) integral polynomial in momenta for 3 sub-Riemannian structures on Carnot groups:

1. *(2,3,5,6)-Problem.* The parabolic sub-Riemannian structure with growth vector (2,3,5,6)
2. *(2,3,5,7)-Problem.* The maximally symmetric sub-Riemannian structure with growth vector (2,3,5,7)
3. *(2,3,5,8)-Problem.* An example of a sub-Riemannian structure on a truncated free graded nilpotent Lie algebra with 2 generators and growth vector (2,3,5,8)

Results

6D example: Degree 6, 7th prolongation step

equations	unknowns	no. of integrals	computation time
$28512 > 12743$	$20790 > 9692$	**130**	43h

7D example: Degree 5, 6th prolongation step

$25872 > 9543$	$16632 > 7139$	**166**	12.3h

8D example: Degree 5, 6th prolongation step

$48048 > 6005$	$28512 > 4686$	**314**	12.5h

All results equal the number of trivial integrals:

$$\Lambda_0^d = \sum_{i=0}^{[d/2]} \binom{d - 2i + D - 3}{D - 3}$$

D: dimension, d: degree

Summary

Computer algebra can be used to determine effectively the number of independent, involutive, smooth integrals polynomial in momenta for Hamiltonian systems.

- The algorithm can prove non-existence if no additional integrals exist
- If there is an additional integral in involution with the others, its degree can be tracked

- Results are rigorous (no numerical approximation)
- The maximal reachable degree is only limited by the available computer strength
- The algorithm is also applicable for parametrized metrics (ongoing work)
- Looking for examples where integrability is expected